$\sum_{n=1}^{\infty} a_n$ K $\Leftrightarrow$ $s_n = a_1 + a_2 + \dots + a_n$

$\exists \lim_{n \to \infty} s_n \in \mathbb{R}$

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nutná podmínka $\sum_{n=1} a_n$ K $\Rightarrow \lim_{n\to\infty} a_n = 0$

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srovnávací krit. $0 \le a_n \le b_n$ $\sum b_n$ K $\Rightarrow \sum a_n$ K

$\sum a_n$ D $\Rightarrow \sum b_n$ D

$\boxed{\text{Věta L5.5}}$ (Cauchyovo odmocninové kritérium)

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy.

(i) $\exists q \in (0,1)\ \exists n_0 \in \mathbb{N}\ \forall n \ge n_0 : \sqrt[n]{a_n} < q \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje

(ii) $\limsup_{n\to\infty} \sqrt[n]{a_n} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje.

(iii) $\lim_{n\to\infty} \sqrt[n]{a_n} < 1. \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje.

(iv) $\limsup \sqrt[n]{a_n} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverguje

(v) $\lim_{n\to\infty} \sqrt[n]{a_n} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverguje

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Poznámky: $\lim_{n\to\infty} \sqrt[n]{a_n} = 1 \Rightarrow$ NEVÍME NIC $\lim \sqrt[n]{n} = 1$

$\sum_{n=1}^{\infty} 1$ D $\lim_{n\to\infty} \sqrt[n]{1} = 1$ | $\sum_{n=1}^{\infty} \frac{1}{n^2}$ K $\lim_{n\to\infty} \sqrt[n]{\frac{1}{n^2}} = \frac{1}{\lim \sqrt[n]{n} \cdot \lim \sqrt[n]{n}} = 1.$

Příklad: $\sum_{n=1}^{\infty} \underbrace{\frac{n^3}{2^n}}_{a_n}$ $\quad \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^3}{2^n}} = \frac{\lim \sqrt[n]{n} \cdot \lim \sqrt[n]{n} \cdot \lim \sqrt[n]{n}}{2} = \frac{1}{2} < 1$ $\quad \boxed{2-2}$

Podle odmocninového kritéria $\sum_{n=1}^{\infty} \frac{n^3}{2^n}$ K.

Dk: (i) $b_n = q^n$ víme $a_n < b_n \; \forall n \geq n_0$ $\quad \underbrace{\sum b_n}_{\sum q^n}$ K $\overset{V5.3.}{\Longrightarrow} \sum_{n=1}^{\infty} a_n$ K.

(i) ⇒ (ii) $\quad b_n = \sup\{\sqrt[n]{a_n}, \sqrt[n+1]{a_{n+1}}, \sqrt[n+2]{a_{n+2}}, \dots\}$

$\lim_{n\to\infty} b_n = \limsup_{n\to\infty} \sqrt[n]{a_n} < 1$. Nalezneme $q \in (\limsup \sqrt[n]{a_n}, 1)$

Z definice $\lim b_n$ pro $\varepsilon = q - \limsup \sqrt[n]{a_n}$ $\quad \exists n_0 \; \forall n \geq n_0$

$b_n < q \Rightarrow$ $\quad$ $\Big(\underset{\varepsilon}{\overset{\limsup \sqrt[n]{a_n}}{|\,\,\,\,|}}\,\underset{q}{\Big)}\,\overset{1}{|}$

$b_{n_0} = \sup\{\sqrt[n_0]{a_{n_0}}, \sqrt[n_0+1]{a_{n_0+1}}, \dots\} < q$

$\Rightarrow \forall n \geq n_0 \; \sqrt[n]{a_n} < q$. Podle (i) tedy $\sum a_n$ K

(iii) ⇒ (ii) $\exists \lim_{n\to\infty} \sqrt[n]{a_n} \Rightarrow \limsup_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n} < 1 \overset{(ii)}{\Longrightarrow} \sum a_n$ K

(iv) $\limsup_{n\to\infty} \sqrt[n]{a_n} > 1$ $\quad \overset{1}{|}\;\; \overset{\,}{\underset{q}{|}}\;\; \underset{\limsup \sqrt[n]{a_n}}{)}$. Analogicky „(i) ⇒ (ii)" $\exists n_0$

$\forall n \geq n_0 \quad b_n > q > 1 \Rightarrow \sup\{\sqrt[n]{a_n}, \sqrt[n+1]{a_{n+1}}, \dots\} > q \Rightarrow$

$\exists n_k \quad \sqrt[n_k]{a_{n_k}} > q > 1 \Rightarrow a_{n_k} > 1 \Rightarrow \lim a_n \neq 0 \overset{V5.1.}{\Longrightarrow} \sum a_n$ D.

(iv) ⇒ (v) $\lim \sqrt[n]{a_n} = \limsup \sqrt[n]{a_n}$

Věta V5.6 (d'Alembertovo podílové kritérium)

Nechť a_n je řada s kladnými členy

(i) $\exists q \in (0,1)\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : \frac{a_{n+1}}{a_n} < q \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje.

(ii) $\limsup\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje

(iii) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ konverguje.

(iv) $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1 \Rightarrow \sum_{n=1}^{\infty} a_n$ diverguje

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Příklad: 1) $\sum_{n=1}^{\infty} \frac{1}{n!}$ $\quad \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim\limits_{n\to\infty} \frac{1}{n+1} = 0 < 1$

podílové krit. $\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n!}$ K

2) $x > 0$ $\quad \sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\quad \lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty} \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} = \lim\limits_{n\to\infty} \frac{x}{n+1} = 0 < 1$

podílové krit. $\Rightarrow \sum_{n=1}^{\infty} \frac{x^n}{n!}$ K

Poznámka: $\sum a_n$ konverguje jako (nebo rychleji než) $\sum q^n \Rightarrow$ V5.5 V5.6 ☺

$\sum a_n \sim \sum \frac{1}{n^\alpha} \Rightarrow \lim \frac{a_{n+1}}{a_n} = 1 = \lim\limits_{n\to\infty} \sqrt[n]{a_n}$ ☹

Dk: (i) Víme $a_{n_0+1} < q \cdot a_{n_0}$

dále $a_{n_0+2} < q \cdot a_{n_0+1} < q^2 \cdot a_{n_0}$

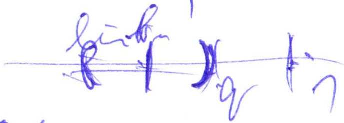
MI $a_{n_0+k} < q^k \cdot a_{n_0}$

$\sum_{k=1}^{\infty} q^k \cdot a_{n_0}$ K $\overset{V2.3.}{\Longrightarrow} \sum_{k=1}^{\infty} a_{n_0+k}$ K

$\Longrightarrow \sum_{n=1}^{\infty} a_n$ K

(i) $\Rightarrow$ (ii) $\quad b_n = \sup\left\{\frac{a_{n+1}}{a_n}, \frac{a_{n+2}}{a_{n+1}}, \dots\right\}$

$\lim_{n\to\infty} b_n = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$. Zvolme $q \in (\lim b_n, 1)$



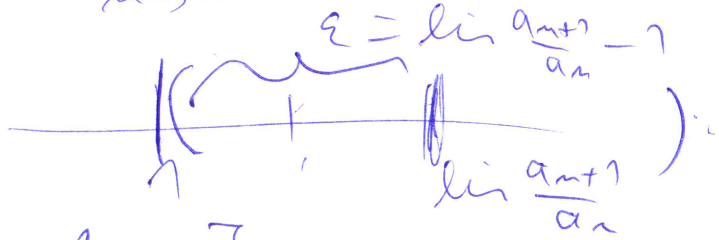
$\exists n_0 \; \forall n \geq n_0 \; b_n < q \quad \sup\left\{\frac{a_{n_0+1}}{a_{n_0}}, \frac{a_{n_0+2}}{a_{n_0+1}}, \dots\right\} < q$

$\Rightarrow \forall n \geq n_0 \; \frac{a_{n+1}}{a_n} < q \overset{(i)}{\Longrightarrow} \sum_{n=1}^{\infty} a_n$ K

(ii) $\Rightarrow$ (iii)

$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1 \Rightarrow \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} a_n < 1 \overset{(ii)}{\Longrightarrow} \sum_{n=1}^{\infty} a_n$ K

(iv) $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$



Z ~~definice~~ definice limity pro $\varepsilon = \lim \frac{a_{n+1}}{a_n} - 1$ $\quad \exists n_0$

$\forall n \geq n_0 \quad \frac{a_{n+1}}{a_n} > 1 \Rightarrow a_{n+1} > a_n$

máme rostoucí posloupnost kladných čísel $\Rightarrow \exists \lim_{n\to\infty} a_n \neq 0 \overset{V5.1.}{\Longrightarrow} \sum_{n=1}^{\infty} a_n$ D $\square$

Věta T 5.7 (kondenzační kritérium)

Nechť $\sum_{n=1}^{\infty} a_n$ je řada s nezápornými členy splňující $a_{n+1} \leq a_n$ $\forall n \in \mathbb{N}$.

Pak $\sum_{n=1}^{\infty} a_n$ konverguje $\Longleftrightarrow$ $\sum_{n=1}^{\infty} 2^n \cdot a_{2^n}$ konverguje.

~~Příklad~~: Nechť $\alpha \in \mathbb{R}$, $\alpha > 0$

Důsledek: (i) $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ konverguje $\Longleftrightarrow$ $\alpha > 1$

$\sum \frac{1}{n}$ D $\qquad \sum \frac{1}{n^2}$ K

$\frac{1}{(n+1)^\alpha} \leq \frac{1}{n^\alpha}$ ✓

geom. řada

Podle kondenzačního kritéria

$$\sum \frac{1}{n^\alpha} \text{ K} \Longleftrightarrow \sum_{n=1}^{\infty} 2^n \cdot \frac{1}{(2^n)^\alpha} = \sum_{n=1}^{\infty} \frac{1}{\left(2^{(\alpha-1)}\right)^n} \text{ K} \Longleftrightarrow \frac{1}{2^{\alpha-1}} < 1$$

$\Longleftrightarrow \alpha > 1$

(ii) $\sum_{n=2}^{\infty} \frac{1}{n \cdot \lg^\alpha n}$ K $\Longleftrightarrow \alpha > 1$

V5.7. $\frac{1}{(n+1) \cdot \lg^\alpha (n+1)} \leq \frac{1}{n \cdot \lg^\alpha n}$

$$\sum_{n=2}^{\infty} \cancel{2^n} \cdot \frac{1}{\cancel{2^n} \cdot \lg^\alpha 2^n} = \sum_{n=2}^{\infty} \frac{1}{n^\alpha \cdot \lg^\alpha 2} \text{ K} \Longleftrightarrow \alpha > 1$$

Dk: Zřejmé

2^{n-1} členů | 2 - 6

$$2^{n-1}\cdot a_{2^{n-1}} \geq s_{2^n} - s_{2^{n-1}} = a_{2^n} + a_{2^n-1} + \dots + a_{2^{n-1}+1} \geq \frac{1}{2} 2^n \cdot a_{2^n}$$

Toto sečteme a dostaneme

$$\sum_{n=1}^{k} 2^{n-1}\cdot a_{2^{n-1}} \geq \underbrace{(s_{2^k} - \cancel{s_{2^{k-1}}}) + (\cancel{s_{2^{k-1}}} - s_{2^{k-2}}) + \dots}_{s_{2^k} - s_0} \geq \left(\sum_{n=1}^{k} 2^n \cdot a_{2^n}\right)\cdot\frac{1}{2}$$

$s_0 = 0$

Víme $a_n \geq 0 \Rightarrow \exists \lim_{n\to\infty} s_n = s \in [0,\infty] \Rightarrow \exists \lim_{n\to\infty} s_{2^k} = s$

"$\Rightarrow$" $\sum \frac{1}{n^\alpha}$ K $\Rightarrow s \in \mathbb{R}$ a $s_{2^k} \leq s$

Dále $s \geq s_{2^k} \geq \left(\sum_{n=1}^{k} 2^n \cdot a_{2^n}\right)\cdot\frac{1}{2} \Rightarrow$

$$\sum_{n=1}^{\infty} 2^n \cdot a_{2^n} \leq 2s \Rightarrow \sum_{n=1}^{\infty} 2^n \cdot a_{2^n} \text{ K}$$

"$\Leftarrow$" $\sum_{n=1}^{\infty} 2^n \cdot a_{2^n}$ K $\Rightarrow \sum_{n=1}^{\infty} 2^{n-1}\cdot a_{2^{n-1}}$ K $\leq S \in [0,\infty]$

Dále $s_{2^k} \leq \sum_{n=1}^{k} 2^{n-1}\cdot a_{2^{n-1}} \leq S \Rightarrow \lim_{n\to\infty} s_{2^k} = s \in \mathbb{R}$

$\Rightarrow \sum a_n$ K $\square$ | $\sum \frac{1}{n} = +\infty$ $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ | $\sum_{n=1}^{\infty} \frac{1}{n^5} \in \mathbb{R}\setminus\mathbb{Q}$? $\mathbb{Q}$?