1) $\lim\limits_{n \to \infty} \int_0^\infty \underbrace{\frac{1}{\left(1+\frac{x}{n}\right)^n \sqrt[n]{x}}}_{f_n(x)} \, dx$

$\underline{\lim\limits_{n\to\infty}} f_n(x) = e^{-x}$ pro $\forall x \in (0,\infty)$

$\left(1+\frac{x}{n}\right)^n \nearrow e^x$

$$\sqrt[n]{x} \begin{cases} \nearrow 1, & x \in (0,1) \\ \searrow 1, & x \in [1,\infty) \end{cases}$$

Položme $g(x) = \left(\chi_{(0,1)}\, x^{\frac{1}{2}} + \chi_{[1,\infty)} \cdot 1\right) \cdot \left(1+\frac{x}{2}\right)^2$

Pak pro $n \geq 2$ $f_n(x) \leq \frac{1}{g(x)}$ na $(0,\infty)$.

$$\int_0^\infty \frac{1}{g(x)}\, dx \overset{g\geq 0}{=} \int_0^1 \frac{1}{g(x)}\, dx + \int_1^\infty \frac{1}{g(x)}\, dx$$

$g \geq 0 \Rightarrow$ (integrály konvergují $\Leftrightarrow$ konvergují jako Newtonovi)

$$\int_0^1 \frac{1}{g(x)}\,dx < \infty \quad \text{podle LSK} \quad (x^{-\frac{1}{2}})$$

$$\int_1^\infty \frac{1}{g(x)}\,dx < \infty \quad \text{podle LSK} \quad (x^{-2})$$

Závěr: $\int_0^\infty \frac{1}{g(x)}\,dx < \infty \Rightarrow$ podle Lebesgueovy

věty je $\lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \int_0^\infty e^{-x}\,dx = 1$.

alternativně k monotonii $\left(1+\frac{x}{n}\right)^n$ lze použít

Bernoulliho nerovnost: $\forall n \, \forall x \geq -2: (1+x)^n \geq 1+nx$

2) $$\lim_{n \to \infty} \int_0^\infty \underbrace{\frac{\log(x+n)}{n} e^{-x}}_{f_n(x)} \, dx$$

$$\lim_{n \to \infty} f_n(x) = 0 \quad \text{pro} \ \forall x \in (0, \infty)$$

$$f_n(x) \leq \left(\frac{x}{n} + 1\right) e^{-x} \leq x e^{-x} + e^{-x}, \quad \int_0^\infty x e^{-x} + e^{-x} \, dx < \infty$$

$\hookrightarrow$ fce ≥ 0 i.e. int konverguje $\Leftrightarrow$ konverguje jako Newtonův, LSL ($e^{\frac{x}{n}}$, Alternativně:

$$(x+n)^{\frac{1}{n}} = \sqrt[n]{n}\left(1+\frac{x}{n}\right)^{\frac{1}{n}} = \sqrt[n]{n}\left(\left(1+\frac{x}{n}\right)^{n}\right)^{\frac{1}{n^2}} \leq$$

$$\leq C\left(1+\frac{x}{n}\right)^{n} \leq C e^{x}$$

$$\frac{\log(x+n)}{n} = \log\left(\exp\left(\frac{\log(x+n)}{n}\right)\right) \leq \log\left(C e^{x}\right) =$$

$$= \log(C) \cancel{+} x, \text{ dál stejně.}$$

Závěr: $\lim\limits_{n \to \infty} \int_0^\infty f_n(x)\, dx = 0$

[Je dobré znát:

$\log(x+y) \leq x+y$ kde $x > 0$, $y > -x$.

$\exp(x+y) \geq x+y$, $x, y \in \mathbb{R}$]

3) $$\lim_{n\to\infty} \int_0^{\pi/4} \frac{1}{\log\left(\frac{n-1}{n} - \sin(x)\right)}\, dx$$

$$\lim_{n\to\infty} f_n(x) = \frac{1}{\log(1-\sin(x))} \quad \text{pro } \forall x \in (0, \tfrac{\pi}{4})$$

$$0 < \sin(x) \overset{x\in(0,\frac{\pi}{4})}{\leq} \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2} < 1$$

$$\exists n_0 \ \forall n \geq n_0: \ \sin(x) \leq \frac{n-1}{n} \text{ na } (0, \tfrac{\pi}{4})$$

Pro $n \geq n_0$ je f_n def. a spojitá na $\left(0, \frac{\pi}{4}\right)$

$\Rightarrow \int_0^{\frac{\pi}{4}} f_n(x)\,dx$ je pro dané n konečný!

$$\frac{n-1}{n} - \sin(x) \nearrow 1 - \sin(x) \Rightarrow \frac{1}{\log\left(\frac{n-1}{n} - \sin(x)\right)}$$

$$\searrow \frac{1}{\log(1-\sin(x))} = f(x).$$

Tedy $-f_n(x) \nearrow -f(x)$ pro $\forall x$ a

$$\int_0^{\pi/4} -f_{n_0}(x)\,dx \in \mathbb{R} \overset{\text{Levi}}{\Longrightarrow} \lim_{n\to\infty} \int_0^{\pi/4} -f_n(x)\,dx = \int_0^{\pi/4} -f(x)\,dx$$

$$\Rightarrow \lim_{n\to\infty} \int_0^{\frac{\pi}{4}} f_n(x)\,dx = \int_0^{\frac{\pi}{4}} f(x)\,dx$$

$-f \geq 0 \Rightarrow \int_0^{\pi/4} -f(x)\,dx < \infty \iff$ konv. jako

Newtonův. Použiji LSK s x^{-1}:

$$\lim_{x \to 0_+} \frac{x}{-\log(1-\sin(x))} = \lim_{x \to 0_+} - \underbrace{\frac{-\sin(x)}{\log(1-\sin(x))}}_{\to 1} \cdot \underbrace{\frac{-x}{-\sin(x)}}_{\to 1} \underbrace{\frac{x}{-x}}_{-1}$$

$= 1.$

Závěr: $\lim\limits_{n \to \infty} \int_0^{\frac{\pi}{4}} \frac{1}{\log\left(\frac{n-1}{n} - \sin(x)\right)} \, dx = -\infty$

4) $\lim\limits_{n\to\infty} \int_0^\infty \underbrace{e^{-x^n}}_{f_n(x)}\, dx$

$\lim\limits_{n\to\infty} f_n(x) = \chi_{(0,1)}(x)$ pro λ-s.v. $x \in (0,\infty)$.

$f_n(x) \leq \chi_{(0,1)}(x) + \chi_{(1,\infty)}\, e^{-x} = g(x)$ pro λ-s.v. $x \in (0,\infty)$

$\int_0^\infty g(x) < \infty, \quad g \geq 0.$

Závěr: $\lim\limits_{n\to\infty} \int_0^\infty e^{-x^2}\, dx = 1$

Alternativně:

$$\int_0^\infty f_n(x)\,dx \overset{f_n\geq 0}{=} \int_1^\infty f_n(x)\,dx + \int_0^1 f_n(x)\,dx$$

$f_n(x) \nearrow f$ na $(0,1)$, $\int_0^\infty f_n(x)\,dx = \int_0^\infty e^{-x}\,dx = 1$

$f_n(x) \searrow f$ na $(1,\infty)$

Levi
$$\Rightarrow \lim_{n\to\infty} \int_0^\infty f_n(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx + \lim_{n\to\infty} \int_1^\infty f_n(x)\,dx = \int_0^\infty f(x)\,dx = 1$$

$[f_n \geq 0]$

5) Nechť f splňuje $\int_{-\infty}^{\infty} |f| \, d\lambda < \infty$

Ukažte, že $\lim_{n \to \infty} \frac{1}{2n} \int_{-n}^{n} f(x) \, dx = 0$

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platí $|f(x)| < \infty$ pro λ-s.v. $x \in \mathbb{R}$

$\Rightarrow \chi_{[-n,n]}(x) \frac{1}{2n} f(x) \to 0$ pro λ-s.v. $x \in \mathbb{R}$

a $|$——— // ———$| \le |f(x)|$

Lebesgue

$$\Rightarrow \quad \lim_{n \to \infty} \frac{1}{2n} \int_{-n}^{n} f(x)\,dx = 0.$$