Introduction to risk modeling, measuring and managing

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Risk Theory – exercises

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Assumption

Probability distribution is known precisely or we have its good estimate:

1. Parametric distribution (multivariate normal, skewed t, ...)
2. Empirical distribution (historical data)
3. (Quasi-)Monte Carlo sample, bootstrap
4. Time series, e.g. garch, VAR
5. ...

2., 3., 4. lead to a discrete distribution.

Markowitz

Markowitz (1952):

$$\min \text{Risk} \;\&\; \max \text{Expected return}$$
$$\text{s.t. portfolio composition constraints}$$

Contents

General deviation measures

Rockafellar et al. (2006A, 2006B): an extension of *standard deviation* which need not to be symmetric with respect to *upside* $X - \mathbb{E}[X]$ and *downside* $\mathbb{E}[X] - X$ of a random variable X.

General deviation measures

Any functional $\mathcal{D} : \mathcal{L}_2(\Omega) \to [0, \infty]$ is called a **general deviation measure** if it satisfies

(D1) shift invariance: $\mathcal{D}(X + C) = \mathcal{D}(X)$ for all X and constants C,

(D2) positive homogeneity: $\mathcal{D}(0) = 0$, and $\mathcal{D}(\lambda X) = \lambda \mathcal{D}(X)$ for all X and all $\lambda > 0$,

(D3) subadditivity: $\mathcal{D}(X + Y) \leq \mathcal{D}(X) + \mathcal{D}(Y)$ for all X and Y,

(D4) nonegativity: $\mathcal{D}(X) \geq 0$ for all X, with $\mathcal{D}(X) > 0$ for nonconstant X.

(D2) & (D3) $\Rightarrow$ convexity

General Deviation Measures

- **Standard deviation**

$$\mathcal{SD}(X) \quad = \quad \sigma(X) = \sqrt{\mathbb{E}\,\|X - \mathbb{E}[X]\|_2}$$

- **Mean absolute deviation**

$$\mathcal{MAD}(X) = \mathbb{E}\big[|X - \mathbb{E}[X]|\big].$$

- **Mean absolute lower and upper semideviation**

$$\mathcal{LSD}_-(X) = \mathbb{E}\big[|X - \mathbb{E}[X]|_-\big],\ \mathcal{USD}_+(X) = \mathbb{E}\big[|X - \mathbb{E}[X]|_+\big].$$

- **CVaR deviation** for $\alpha \in (0, 1)$:

$$\mathcal{D}_\alpha(X) = \min_{\xi\in\mathbb{R}} \frac{1}{1-\alpha}\mathbb{E}[\max\{(1-\alpha)(X-\xi), \alpha(\xi - X)\}].$$

Coherent risk and return measures

Artzner et al. (1999): $\mathcal{R} : \mathcal{L}_2(\Omega) \to (-\infty, \infty]$ that satisfies

(R1) shift equivariance: $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C,

(R2) positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$,

(R3) subadditivity: $\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$ for all X and Y,

(R4) monotonicity: $\mathcal{R}(X) \leq \mathcal{R}(Y)$ when $X \geq Y$ a.s..

Coherent risk measures

CVaR for $\alpha \in (0,1)$:

$$CVaR_\alpha(X) = \min_{\xi \in \mathbb{R}} \xi + \frac{1}{1-\alpha} \mathbb{E}[\max\{(-X-\xi), 0\}]$$

Monotonicity (R4)

$$X \geq Y \text{ a.s.} \;\Rightarrow\; \mathcal{R}(X) \leq \mathcal{R}(Y).$$

"Higher gain (almost sure), lower risk."

Subadditivity (R3), (D3)

$$\mathcal{R}(X+Y) \leq \mathcal{R}(X) + \mathcal{R}(Y).$$

"Holding two assets together is never more risky than holding them separately $\leftrightarrow$ diversification."

Positive homogeneity (R2), (D2)

For all X and all $\lambda > 0$

$$\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X).$$

"Increasing our position $\lambda-$times increases the risk proportionally."

Convexity

The axioms

(R2) positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$,

(R3) subadditivity: $\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$ for all X and Y,

imply **convexity**: for arbitrary $\lambda \in (0, 1)$ and X, Y

$$\mathcal{R}(\lambda X + (1-\lambda)Y) \leq \mathcal{R}(\lambda X) + \mathcal{R}((1-\lambda)Y) \leq \lambda \mathcal{R}(X) + (1-\lambda)\mathcal{R}(Y).$$

Translation invariance vs. equivariance

For a constant C

- Shift invariance: $\mathcal{R}(X + C) = \mathcal{R}(X) - C$
- Shift equivariance: $\mathcal{D}(X + C) = \mathcal{D}(X)$

"Sure gain decreases risk OR leaves it unchanged."

Additional properties

We say that general deviation measure $\mathcal{D}$ is

(LSC) **lower semicontinuous** (lsc) if all the subsets of $\mathcal{L}_2(\Omega)$ having the form $\{X : \mathcal{D}(X) \leq c\}$ for $c \in \mathbb{R}$ (level sets) are closed;

(D5) **lower range dominated** if $\mathcal{D}(X) \leq EX - \inf_{\omega \in \Omega} X(\omega)$ for all X.

Strictly expectation bounded risk measures satisfy (R1), (R2), (R3), and

(R5) $\mathcal{R}(X) > \mathbb{E}[-X]$ for all nonconstant X, whereas $\mathcal{R}(X) = \mathbb{E}[-X]$ for constant X.

Strictly expectation bounded risk measures

Theorem 2 in Rockafellar et al (2006 A):

Theorem

Deviation measures correspond one-to-one with strictly expectation bounded risk measures under the relations

- $\mathcal{D}(X) = \mathcal{R}(X - \mathbb{E}[X])$
- $\mathcal{R}(X) = \mathbb{E}[-X] + \mathcal{D}(X)$

In this correspondence, $\mathcal{R}$ is coherent if and only if $\mathcal{D}$ is lower range dominated.

Mean absolute deviation from $(1-\alpha)$-th quantile

CVaR deviation

For any $\alpha \in (0,1)$ a finite, continuous, lower range dominated deviation measure

$$\mathcal{D}_\alpha(X) = CVaR_\alpha(X - \mathbb{E}[X]). \tag{1}$$

The deviation is also called **weighted mean absolute deviation from the** $(1-\alpha)$-th **quantile**, see Ogryczak, Ruszczynski (2002), because it can be expressed as

$$\mathcal{D}_\alpha(X) = \min_{\xi \in \mathbb{R}} \frac{1}{1-\alpha} \mathbb{E}[\max\{(1-\alpha)(X-\xi), \alpha(\xi - X)\}] \tag{2}$$

with the minimum attained at any $(1-\alpha)$-th quantile. In relation with CVaR minimization formula, see Pflug (2000), Rockafellar and Uryasev (2000, 2002).

General deviation measures

If $\mathcal{D}_0, \mathcal{D}_1, \dots, \mathcal{D}_K$ are general deviation measures, then

- $\mathcal{D} = \lambda \mathcal{D}_0$ for $\lambda > 0$,
- $\mathcal{D} = \max\{\mathcal{D}_1, \dots, \mathcal{D}_K\}$,
- $\mathcal{D} = \lambda_1 \mathcal{D}_1 + \dots + \lambda_K \mathcal{D}_K$, for $\lambda_k > 0$ and $\sum_{k=1}^K \lambda_k = 1$.

are general deviation measures too.

Proposition 4, Rockafellar et al (2006 A):

Example – Variance

- Variance is not coherent risk measure, nor general deviation measure.
- Standard deviation is a general deviation measure.
- $$\mathcal{SD}(X) - \mathbb{E}[X]$$
  is a coherent risk measure.

Contents

Value at Risk, Conditional Value at Risk

Value at Risk the losses lower or equal to VaR appear with a high probability α and the losses higher than VaR apper with low probability $1 - \alpha$.

Conditional Value at Risk the expected value of $(1 - \alpha)$*100% worst losses (not always the same as the expected value of the losses higher than/higher or equal to Value at Risk)

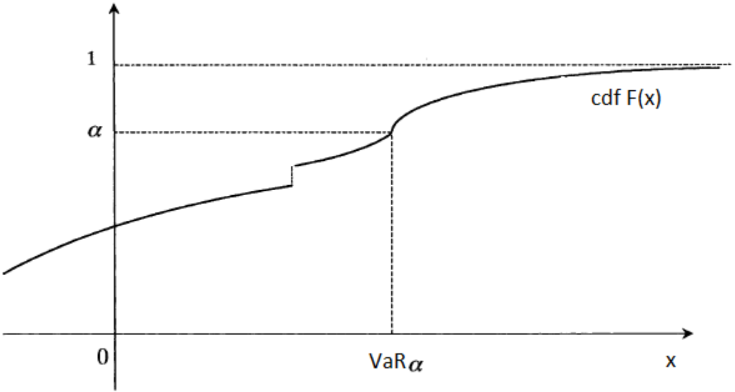
Value at Risk (VaR)

Value at Risk (VaR) for a general **loss random variable** Z defined on probability space $(\Omega, \mathcal{A}, P)$, level $\alpha \in (0,1)$, usually 0.95, 0.99, 0.995:

$$VaR_\alpha(Z) = \min_z z \text{ s.t. } P(Z \le z) \ge \alpha.$$

Upper Value at Risk (upper-VaR)

$$VaR_\alpha^+(Z) = \inf_z z \text{ s.t. } P(Z \le z) > \alpha.$$


1
α
cdf F(x)
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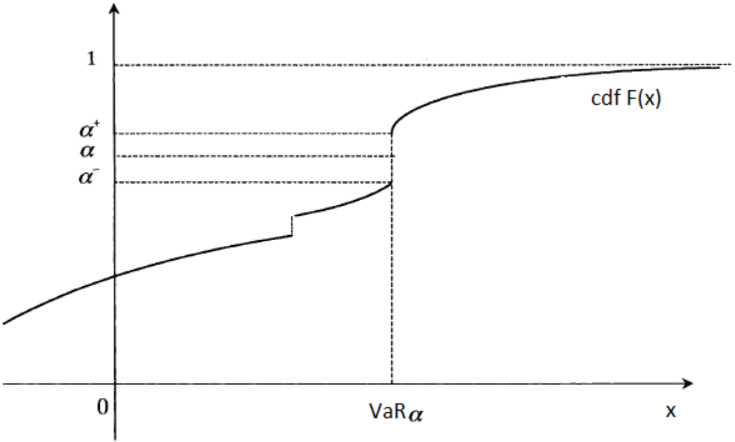
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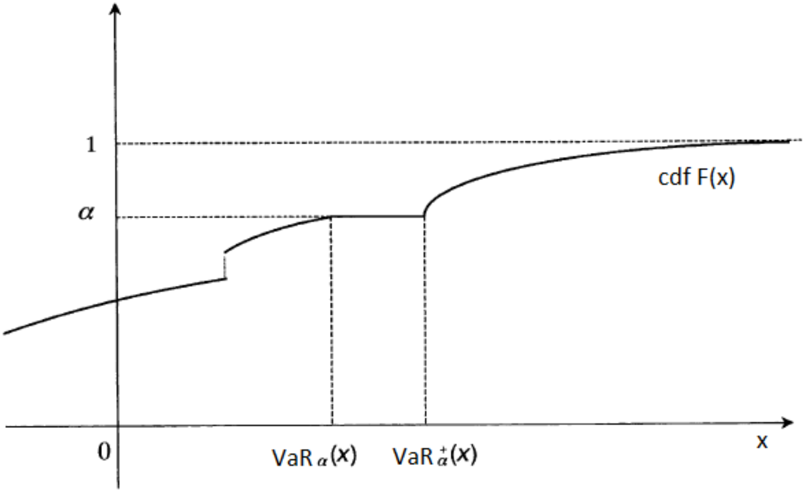
1
α
cdf F(x)
0
VaR α(x)
VaR +α(x)
x

VaR under discrete distribution

Let Z be concentrated in finitely many points $z^{[1]} < z^{[2]} < \cdots < z^{[N]}$ with probabilities $P\big(Z = z^{[k]}\big) = p^{[k]} > 0$, $\sum_{k=1}^{N} p^{[k]} = 1$.
Find index k_α such that

$$\sum_{k=1}^{k_\alpha - 1} p^{[k]} < \alpha \leq \sum_{k=1}^{k_\alpha} p^{[k]}.$$

Then we have

$$VaR_\alpha(x) = z^{[k_\alpha]}. \tag{3}$$

Value at Risk – axioms

Value at Risk fulfills

(R1) shift equivariance: $\mathcal{R}(X + C) = \mathcal{R}(X) - C$ for all X and constants C,

(R2) positive homogeneity: $\mathcal{R}(0) = 0$, and $\mathcal{R}(\lambda X) = \lambda \mathcal{R}(X)$ for all X and all $\lambda > 0$,

(R4) monotonicity: $\mathcal{R}(X) \leq \mathcal{R}(Y)$ when $X \geq Y$.

However, in general, **it does not fulfill**

(R3) subadditivity: $\mathcal{R}(X + Y) \leq \mathcal{R}(X) + \mathcal{R}(Y)$ for all X and Y.

Example

Two independent one-year bonds with nominal value 1 CZK and the same parameters

- No loss with probability 96%, loss 0.7 with probability 4%, thus Value at Risk on the level 95% is equal to 0.
- If you buy both bonds, then we have the following losses and probabilities
  - 0 with probability 92.16% (= 0.96 * 0.96)
  - 0.7 with prob. 7.68% (= 2 * 0.96 * 0.04)
  - 1.4 with prob. 0.16% (=0.04 * 0.04)

  Thus Value at Risk of $Z_1 + Z_2$ is 0.7, i.e.

$$\mathrm{VaR}_{0.95}(Z_1 + Z_2) > \mathrm{VaR}_{0.95}(Z_1) + \mathrm{VaR}_{0.95}(Z_2).$$

Example – consequences

- Value at Risk is not subadditive

$$\mathrm{VaR}_{0.95}(Z_1 + Z_2) > \mathrm{VaR}_{0.95}(Z_1) + \mathrm{VaR}_{0.95}(Z_2).$$

- Even for independent losses (risks) it holds

$$\mathrm{VaR}_{0.95}(Z_1 + Z_2) \neq \mathrm{VaR}_{0.95}(Z_1) + \mathrm{VaR}_{0.95}(Z_2).$$

Contents

Conditional Value at Risk (CVaR)

For $Z \in \mathcal{L}_1(\Omega)$, **Conditional Value at Risk** (CVaR) is defined as the mean of losses in the α-tail distribution with the distribution function:

$$F_\alpha(\eta) = \begin{cases} \frac{F(\eta)-\alpha}{1-\alpha}, & \text{if } \eta \geq \mathrm{VaR}_\alpha(Z), \\ 0, & \text{otherwise}, \end{cases}$$

where $F(\eta) = P(Z \leq \eta)$.

Example

Two independent one-year bonds with nominal value 1 CZK and the same parameters

- No loss with probability 96%,
- loss 0.7 with probability 4%,

thus Value at Risk on the level 95% is equal to 0.

$$
\begin{aligned}
\mathrm{CVaR}^{+}_{0.95}(Z_1) &= \mathbb{E}[Z_1 | Z_1 > \mathrm{VaR}_{0.95}(Z_1)] \\
&= \frac{1}{0.04}(0.04 \cdot 0.7) = 0.7 \\
\mathrm{CVaR}^{-}_{0.95}(Z_1) &= \mathbb{E}[Z_1 | Z_1 \geq \mathrm{VaR}_{0.95}(Z_1)] \\
&= \frac{1}{0.96 + 0.04}(0.96 \cdot 0 + 0.04 \cdot 0.7) = 0.028
\end{aligned}
$$

Let Z be concentrated in finitely many points $z^{[1]} < z^{[2]} < \cdots < z^{[N]}$ with probabilities $P(Z = z^{[k]}) = p^{[k]} > 0$, $\sum_{k=1}^{N} p^{[k]} = 1$.
Find index k_α such that

$$\sum_{k=1}^{k_\alpha - 1} p^{[k]} < \alpha \leq \sum_{k=1}^{k_\alpha} p^{[k]}.$$

Then we have

$$VaR_\alpha(x) = z^{[k_\alpha]} \tag{4}$$

and if $\alpha > 1 - p^{[N]}$, then

$$VaR_\alpha(x) = CVaR_\alpha(x) = z^{[N]}, \tag{5}$$

else

$$CVaR_\alpha(x) = \frac{1}{1-\alpha}\left[\left(\sum_{k=1}^{k_\alpha} p^{[k]} - \alpha\right) z^{[k_\alpha]} + \sum_{k=k_\alpha+1}^{N} p^{[k]} z^{[k]}\right]. \tag{6}$$

Example

$$
\begin{aligned}
\mathrm{CVaR}^{+}_{0.95}(Z_1) &= \mathbb{E}[Z_1 | Z_1 > \mathrm{VaR}_{0.95}(Z_1)] \\
&= \frac{1}{0.04}(0.04 \cdot 0.7) = 0.7 \\
\mathrm{CVaR}^{-}_{0.95}(Z_1) &= \mathbb{E}[Z_1 | Z_1 \geq \mathrm{VaR}_{0.95}(Z_1)] \\
&= \frac{1}{0.96 + 0.04}(0.96 \cdot 0 + 0.04 \cdot 0.7) = 0.028
\end{aligned}
$$

$$
\mathrm{CVaR}_{0.95}(Z_1) = \frac{1}{1 - 0.95}\Big((0.96 - 0.95) \cdot 0 + 0.04 \cdot 0.7\Big) = 0.56
$$

Obviously

$$
\mathrm{CVaR}^{-}_{0.95}(Z_1) < \mathrm{CVaR}_{0.95}(Z_1) < \mathrm{CVaR}^{+}_{0.95}(Z_1).
$$

VaR & CVaR

CVaR can be expressed using the following **minimization formula**:

$$\mathrm{CVaR}_\alpha(Z) = \min_{\xi \in \mathbb{R}} \left[\xi + \frac{1}{1-\alpha} \mathbb{E}\left[\max\{Z - \xi, 0\}\right] \right] \quad (7)$$

with the minimum attained at any $(1-\alpha)$-th quantile.

CVaR – coherence

CVaR is a coherent risk measure.

VaR and CVaR under normal distribution

Let $Z \sim N(\mu, \sigma^2)$, then

$$\begin{aligned} \mathrm{VaR}_\alpha(Z) &= \mu + z_\alpha \sigma, & (8) \\ \mathrm{CVaR}_\alpha(Z) &= \mu + \eta_\alpha \sigma, & (9) \end{aligned}$$

where $z_\alpha = \Phi^{-1}(\alpha)$ is a quantile of a standard normal distribution (with pdf ϕ and cdf Φ) and

$$\eta_\alpha = \frac{\int_{\Phi^{-1}(\alpha)}^{\infty} t\phi(t)\,dt}{1-\alpha}.$$

Coherent risk measures.

VaR under normal distribution

Let $Z \sim N(\mu, \sigma^2)$, then

$$\mathrm{VaR}_\alpha(Z) \quad = \quad \mu + z_\alpha \sigma, \tag{10}$$

$$P(Z \leq VaR_\alpha) = P\left(\frac{Z-\mu}{\sigma} \leq \frac{VaR_\alpha - \mu}{\sigma}\right) = \Phi\left(\frac{VaR_\alpha - \mu}{\sigma}\right) = \alpha$$

VaR under normal distribution

Let $Z \sim N(\mu, \sigma^2)$, then

$$\mathrm{CVaR}_\alpha(Z) \;=\; \mu + \eta_\alpha \sigma, \tag{11}$$

$$\begin{aligned}
\mathrm{CVaR}_\alpha(Z) &= \frac{1}{1-\alpha} \int_{\mu+\Phi^{-1}(\alpha)\sigma}^{\infty} \frac{z}{\sigma}\, \phi\left(\frac{z-\mu}{\sigma}\right) dz \\
&= \frac{1}{1-\alpha} \int_{\Phi^{-1}(\alpha)}^{\infty} \frac{\mu + t\sigma}{\sigma} \phi\,(t)\, \sigma dt \\
&= \frac{1}{1-\alpha} \left(\mu \int_{\Phi^{-1}(\alpha)}^{\infty} \phi\,(t)\, dt + \sigma \int_{\Phi^{-1}(\alpha)}^{\infty} t\phi\,(t)\, dt \right)
\end{aligned}$$

Portfolio VaR and CVaR under normal distribution

For a portfolio $R^\top x$ with random vector of returns $R \sim N_n(\mu, Q)$

$$\mathrm{VaR}_\alpha(-R^\top x) = -\mu^\top x + \zeta_\alpha \sqrt{x^\top Q x}, \qquad (12)$$

$$\mathrm{CVaR}_\alpha(-R^\top x) = -\mu^\top x + \eta_\alpha \sqrt{x^\top Q x}. \qquad (13)$$

Table: Quantiles and generalized quantiles

	$c_\beta \diagdown \beta$	0.9	0.95	0.99
VaR	ζ_β	1.2816	1.6449	2.3263
CVaR	η_β	1.7550	2.0627	2.6652

Contents

Portfolio weights

$$\mathcal{X} = \{x : \sum_{i=1}^{x} x_i = 1, \ x_i \geq 0\}.$$

Multiobjective optimization

Denote by

- $\mathbb{E}(x)$ portfolio expected return,
- $\mathcal{R}(x)$ portfolio risk.

$$\begin{aligned} \min \mathcal{R}(x) \ \& \ \max \mathbb{E}(x) \\ \text{s.t. } x \in \mathcal{X}. \end{aligned}$$

OR

$$\begin{aligned} \min \mathcal{R}(x) \ \& \ \min -\mathbb{E}(x) \\ \text{s.t. } x \in \mathcal{X}. \end{aligned}$$

Multiobjective optimization – efficient solutions

We say that portfolio $x \in \mathcal{X}$ is efficient if there is no other portfolio $\tilde{x} \in \mathcal{X}$ such that $\mathbb{E}(x) \leq \mathbb{E}(\tilde{x})$ and $\mathcal{R}(x) \geq \mathcal{R}(\tilde{x})$ with at least one inequality strict.

Multiobjective optimization – efficient solutions

Two basic approaches:

- Aggregate function approach:

$$\begin{aligned} \min \quad & \mathcal{R}(x) - \lambda \mathbb{E}(x) \\ \text{s.t.} \quad & x \in \mathcal{X}. \end{aligned}$$

  for some $\lambda > 0$.

- $\varepsilon-$constrained approach:

$$\begin{aligned} \min \quad & \mathcal{R}(x) \\ \text{s.t.} \quad & \mathbb{E}(x) \geq \varepsilon_E, \\ & x \in \mathcal{X}, \end{aligned}$$

  OR

$$\begin{aligned} \max \quad & \mathbb{E}(x) \\ \text{s.t.} \quad & \mathcal{R}(x) \leq \varepsilon_R, \\ & x \in \mathcal{X}. \end{aligned}$$

Portfolio random loss

Consider n assets with random rate of return R_i

$$Z(x) = -\sum_{i=1}^{n} x_i R_i$$

Investment problem with VaR

Solve a simple investment problem

$$\min_{x_i} \mathrm{VaR}_\alpha \left(-\sum_{i=1}^{n} x_i R_i \right)$$
$$\text{s.t. } \mathbb{E} \left[\sum_{i=1}^{n} x_i R_i \right] \geq r_0,$$
$$\sum_{i=1}^{n} x_i = 1, \; x_i \geq 0.$$

The first constraint ensures minimal expected return r_0, x_i are (nonnegative) portfolio weights which sum to one.

Chance constrained problems – single random constraint

Let $f, g(\cdot, \xi) : \mathbb{R}^n \to \mathbb{R}$ be real functions, $X \subseteq \mathbb{R}^n$, ξ be a real random vector, $\varepsilon \in (0, 1)$ small:

$$\begin{aligned} &\min_{x \in X} f(x) \\ &\quad \text{s.t.} \qquad P\left(g(x, \xi) \le 0\right) \ge 1 - \varepsilon. \end{aligned}$$

INTERPRETATION: for a given $x \in X$, the probability of ξ for which the random constraint is fulfilled must be at least $1 - \varepsilon$:

$$P\left(g(x, \xi) \le 0\right) = P\left(\{\xi : \; g(x, \xi) \le 0\}\right).$$

Chance constrained problems – single random constraint

Let ξ have a finite discrete distribution with realizations $\xi^1, \ldots, \xi^S$ and probabilities $p_s > 0$, $\sum_{s=1}^S p_s = 1$:

$$\begin{aligned} \min_{x,y} f(x) & \\ \text{s.t.} & \\ \sum_{s=1}^S p_s y_s &\geq 1 - \varepsilon, \\ g(x, \xi_s) &\leq M(1 - y_s),\ s = 1, \ldots, S \\ y_s &\in \{0, 1\},\ s = 1, \ldots, S, \\ x &\in X, \end{aligned} \tag{14}$$

where $M \geq \max_{s=1,\ldots,S} \sup_{x \in X} g(x, \xi_s)$.

Value at Risk (VaR)

Portfolio optimization problem:

$$
\begin{aligned}
& \min_{z,x} z \\
P\left(-\sum_{i=1}^{n} R_i x_i \leq z \right) & \geq \alpha, \\
\sum_{i=1}^{n} \mathbb{E}[R_i] \cdot x_i & \geq r_{min}, \\
\sum_{i=1}^{n} x_i = 1,\ x_i & \geq 0,
\end{aligned}
$$

where R_i is random rate of return of $i-$th asset and minimal expected return r_{min} is selected in such way that the problem is feasible.

Homework 2

1. Rewrite the VaR minimization problem under a finite discrete distribution as a mixed-integer LP problem.
2. Use the same dataset as for the CVaR homework, i.e. at least 6 assets, but the number of scenarios is limited to 50 (if you have free GAMS, otherwise you can use all 100 returns).
3. Consider $\alpha = 0.95$ and run the problem for different 11 values $r_0 \in \{\min_i \overline{R}_i, \ldots, \max_i \overline{R}_i\}$.
4. Plot the optimal values VaR_α against the corresponding values of r_0.

If the distribution of R_i is discrete with realizations r_{is} and probabilities $p_s = 1/S$, then we can use **linear programming** reformulation

$$\begin{aligned}
\min_{\xi, x_i, y_s} \ & \xi \\
\text{s.t. } & \frac{1}{S} \sum_{s=1}^{S} y_s \geq \alpha, \\
& -\sum_{i=1}^{n} x_i r_{is} - \xi \leq M(1 - y_s), \ s = 1, \ldots, S, \\
& \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0, \\
& \sum_{i=1}^{n} x_i = 1, \ x_i \geq 0, \\
& \xi \in \mathbb{R}, \ y_s \in \{0, 1\},
\end{aligned}$$

where $\overline{R}_i = 1/S \sum_{s=1}^{S} r_{is}$.

Investment problem with CVaR

Solve a simple investment problem

$$\min_{x_i} \mathrm{CVaR}_\alpha \left(-\sum_{i=1}^n x_i R_i \right)$$

$$\text{s.t. } \mathbb{E}\left[\sum_{i=1}^n x_i R_i \right] \geq r_0,$$

$$\sum_{i=1}^n x_i = 1, \; x_i \geq 0.$$

The first constraint ensures minimal expected return r_0, x_i are (nonnegative) portfolio weights which sum to one.

If the distribution of R_i is discrete with realizations r_{is} and probabilities $p_s = 1/S$, then we can use **linear programming** reformulation

$$\begin{aligned}
\min_{\xi, x_i, u_s}\ & \xi + \frac{1}{(1-\alpha)S}\sum_{s=1}^{S} u_s, \\
\text{s.t. } & u_s \geq -\sum_{i=1}^{n} x_i r_{is} - \xi,\ \ s = 1, \dots, S, \\
& \sum_{i=1}^{n} x_i \overline{R}_i \geq r_0, \\
& \sum_{i=1}^{n} x_i = 1,\ \ x_i \geq 0, \\
& \xi \in \mathbb{R},\ \ u_s \geq 0,
\end{aligned}$$

where $\overline{R}_i = 1/S \sum_{s=1}^{S} r_{is}$.

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