1) Spočtěte $\int_{\Pi} f(x,y,z)\, d\lambda^3(x,y,z)$

kde $f(x,y,z) = \sqrt{x^2+y^2+z^2}$, $\Pi = \{(x,y,z) \in \mathbb{R}^3;\ x^2+y^2+z^2 \leq z\}$

Π je uz. $\Rightarrow$ je měř.

f je spojitá $\Rightarrow$ je měř.

(*) $\varphi : (0,\infty) \times (-\pi, \pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \longrightarrow \mathbb{R}^3$

$$x = r\cos(\alpha)\cos(\beta)$$

$$y = r\sin(\alpha)\cos(\beta)$$

$$z = r\sin(\beta)$$

$$|J\varphi(r,\alpha,\beta)| = r^2\cos\beta$$

$$\varphi^{-1}(M) = \{(r,\alpha,\beta) \in (0,\infty) \times (-\pi,\pi) \times (-\frac{\pi}{2}, \frac{\pi}{2}) \; ; \; r \leq \sin\beta, \; \beta \in (0, \frac{\pi}{2})\}$$

φ splňuje předpoklady.

$$\int_M f(x,y,z)\, d\lambda^3(x,y,z) \overset{(*)}{=} \int_{\varphi^{-1}(M)} r^3 \cos(\beta)\, d\lambda^3(r,\alpha,\beta)$$

$$\overset{\substack{\text{Fubini} \\ \text{Integrand} \geq 0}}{=} \int_0^{\frac{\pi}{2}} \int_0^{\sin(\beta)} \int_{-\pi}^{\pi} r^3 \cos(\beta)\, d\alpha\, dr\, d\beta =$$

$$= 2\pi \int_0^{\pi/2} \cos(\beta) \sin^4(\beta) \cdot \frac{1}{4}\, d\beta =$$

$$= \left| \begin{array}{l} t = \sin(\varphi) \\ dt = \cos(\varphi)\, d\varphi \\ 0 \mapsto 0 \\ \frac{\pi}{2} \mapsto 1 \end{array} \right| = \frac{\pi}{2} \int_0^1 t^4 \, dt = \frac{\pi}{10}$$

2) Spočtěte $\int_\Omega f(x,y,z)\, d\lambda^3(x,y,z)$, kde

$$f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \quad \Omega = \left\{(x,y,z) \in \mathbb{R}^3;\ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z}{c^2} \leq 1\right\}$$

Ω je uz. $\Rightarrow$ je měř.

f je spoj. $\Rightarrow$ je měř.

(*) $$\left.\begin{aligned} x &= a\, r \cos(\alpha) \cos(\beta) \\ y &= b\, r \sin(\alpha) \cos(\beta) \\ z &= c\, r \sin(\beta) \end{aligned}\right\} \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = r^2$$

$$\varphi : \underbrace{(0, r) \times (-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} \to \mathbb{R}^3$$

$$\varphi^{-1}(M) = \{ (r, \alpha, \beta) \in \quad ; \; r \leq 1 \}$$

$$| J\varphi(r, \alpha, \beta) | = a b c\, r^2 \cos\beta$$

φ splňuje předpoklady.

$$\int_M f(x,y,z)\, d\lambda^3(x,y,z) \overset{(*)}{=} \int_{\varphi^{-1}(M)} r^2 \cdot abc\, r^2 \cos(\beta)\, d\lambda^3(r,\alpha,\beta)$$

Fubiní
integrand nemění znaménko

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 \int_{-\pi}^{\pi} abc\, r^4 \cos(\beta)\, d\alpha\, dr\, d\beta =$$

$$= 2\pi\, abc\, \frac{1}{5} \cdot 2 = \frac{4}{5}\, abc\, \pi.$$

Idea:

Mám integrál tvaru

$$\int_0^\infty \frac{f(ax) - f(bx)}{x}\,dx$$

Pokud $f \in C^1((0,\infty))$ a existují limity v ∞ a 0 pak lze dokázat následující.

(Předpokládáme $0 < a < b$)

Uvažme $\int_{(0,\infty)\times(a,b)} -f'(xy)\, d\lambda^2(x,y) = I$

Pokud lze aplikovat Fubiniho větu, dostaneme

$$I = \int_0^\infty \int_a^b -f'(xy)\, dy\, dx =$$

$$= \int_0^\infty \left[-\frac{f(xy)}{x} \right]_{y=a}^{b} dx = \int_0^\infty \frac{f(ax)-f(bx)}{x}\, dx$$

a zároveň

$$I = \int_a^b \int_0^\infty -f'(xy)\, dx\, dy =$$

$$= \int_a^b \left[-\frac{f(xy)}{y} \right]_{x=0}^{\infty} dy =$$

$$= \left(f(0) - \lim_{x\to\infty} f(x) \right) \underbrace{\int_a^b \frac{1}{y}\, dy}_{\log\left(\frac{b}{a}\right)}$$

3) $$\int_0^\infty \frac{\operatorname{arctg}(ax) - \operatorname{arctg}(bx)}{x}\, dx \quad \text{kde } a.b > 0$$

Uvážím pro $a < b$, $a.b > 0$

$$\int -\frac{1}{1+(xy)^2}\, d\lambda^2(x,y) = I$$

$(0,\infty) \times (a,b)$

Integrand nemění znaménko $\Rightarrow$ lze použít Fubiniho věta.

$$I = \int_0^\infty \frac{\operatorname{arctg}(ax) - \operatorname{arctg}(bx)}{x}\, dx$$

$$I = \int_a^b \left[- \frac{\operatorname{arctg}(xy)}{y} \right]_0^\infty dy =$$

$$= \int_a^b - \operatorname{sgn}(y) \frac{\pi}{2} \cdot \frac{1}{y}\, dy = - \operatorname{sgn}(a) \frac{\pi}{2} \int_a^b \frac{1}{y}\, dy$$

$$= - \operatorname{sgn}(a) \frac{\pi}{2} \log\left(\frac{b}{a}\right).$$

Je patrní, že rovnost platí i pro $b \leq a$.

4) Spočtěte $\int_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x^2}\,dx$, kde $a,b\geq 0$

$$\int_a^b e^{-x^2y}\,dy = \left[\frac{e^{-x^2y}}{-x^2}\right]_{y=a}^{b} = \frac{e^{-x^2a}-e^{-x^2b}}{x^2},$$

$e^{-x^2y} > 0$. Takže pro $a<b$:

$$\int_{(0,\infty)\times(a,b)} e^{-x^2y}\,d\lambda^2(x,y) = I$$

$$I \overset{\text{Fubini}}{=} \int_0^\infty \frac{e^{-x^2 a} - e^{-x^2 b}}{x^2}\, dx$$

$$I \overset{\text{Fubini}}{=} \int_a^b \int_0^\infty e^{-x^2 y}\, dx\, dy = \left| \begin{array}{l} t = x\sqrt{y} \\ dt = \sqrt{y}\, dx \\ 0 \mapsto 0 \\ \infty \mapsto \infty \end{array} \right| =$$

$$= \int_a^b \frac{1}{\sqrt{y}} \int_0^\infty e^{-t^2}\, dx\, dy = \frac{\sqrt{\pi}}{2} \int_a^b \frac{1}{\sqrt{y}}\, dy =$$

$$= \frac{\sqrt{\pi}}{2} \left[2\sqrt{y} \right]_a^b = \sqrt{\pi}\left(\sqrt{b} - \sqrt{a}\right).$$

5) $\displaystyle\int_0^\infty \frac{e^{-ax^2}-e^{-bx^2}}{x}\,dx$, $\quad a,b>0$

$$\int_a^b e^{-yx^2}x\,dy = \left[\frac{e^{-yx^2}}{-x}\right]_a^b = \frac{e^{-ax^2}-e^{-bx^2}}{x}$$

Zbytek obdobný.

$$\int_0^\infty \frac{\log(1+x^2a^2) - \log(1+x^2b^2)}{x^2}\,dx, \quad \text{kde } a,b \in \mathbb{R}$$

$$-\frac{\log(1+x^2y^2)}{x^2} \xmapsto{\frac{\partial}{\partial y}} -\frac{1}{1+x^2y^2}\,2y = g(x,y)$$

Nechť $0 \leq a < b$. Pak g nemění znaménko a platí

$$\int_{(0,\infty)\times(a,b)} g(x,y)\,d\lambda^2(x,y) = I$$

$$I \overset{\text{Fubini}}{=} \int_0^\infty \frac{\log(1+a^2x^2) - \log(1+b^2x^2)}{x^2}\, dx$$

$$I \overset{\text{Fubini}}{=} \int_a^b -2 \int_0^\infty \frac{1}{1+x^2y^2}\, y \, dx\, dy = \left| \begin{array}{l} t = xy \\ dt = y\, dx \\ 0 \mapsto 0 \\ \infty \mapsto \operatorname{sgn}(y)\infty \end{array} \right|$$

$$= \int_a^b -2 \Big[\arctan(t) \Big]_{t=0}^{\operatorname{sgn}(y)\infty} dy$$

$$= \int_a^b -\pi \,\mathrm{sgn}(y)\, dy = -\pi(|b| - |a|)$$

Patrně platí i pro $0 \leq b < a$ a pro $a < b \leq 0$, $b < a \leq 0$.

Nechť $a < 0 < b$, pak pro $c = 0$ je

$$\int_0^\infty \frac{\log(1+x^2a^2) - \log(1+x^2b^2)}{x^2}\, dx \overset{!}{=} \int_0^\infty \frac{\log(1+x^2a^2) - \log(1+x^2c^2)}{x^2} +$$

$$+\int_0^\infty \frac{\log(1+x^2c^2) - \log(1+x^2b^2)}{x^2}\,dx =$$

$$= -\pi(-|a|) + (-\pi)(|b|) = -\pi(|b| - |a|)$$

pro $b < 0 < a$ obdobně.