**Věta 4.9** $f, g$ spojité na $[a,b]$, $\exists f'$ na $(a,b)$ $\exists g'$ vlastní, nenulová na $(a,b)$. Pak $\exists\, \xi \in (a,b)$

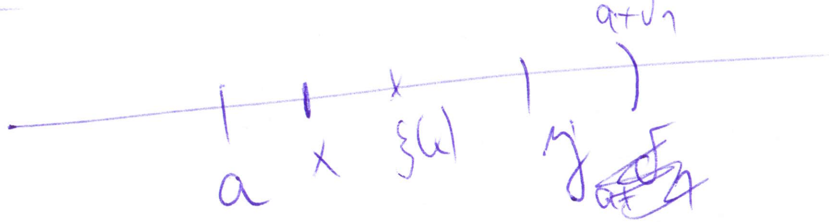
$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

**Věta 4.10** (i) $a \in \mathbb{R}^*$, $\lim_{x\to a_+} f(x) = 0 = \lim_{x\to a_+} g(x) = 0$ a nechť existuje $\lim_{x\to a_+} \frac{f'(x)}{g'(x)}$. Pak $\lim_{x\to a_+} \frac{f(x)}{g(x)} = \lim_{x\to a_+} \frac{f'(x)}{g'(x)}$.

(ii) $a \in \mathbb{R}^*$, $\lim_{x\to a_+} |g(x)| = +\infty$ a nechť existuje $\lim_{x\to a_+} \frac{f'(x)}{g'(x)}$.

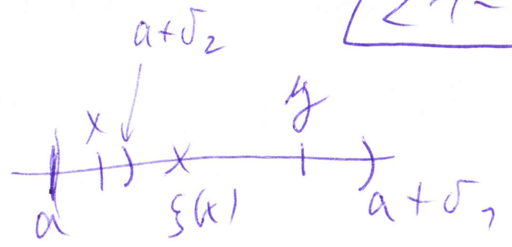
Pak $\lim_{x\to a_+} \frac{f(x)}{g(x)} = \lim_{x\to a_+} \frac{f'(x)}{g'(x)}$.

Idea dk.:




$$\frac{f'(\xi(x))}{g'(\xi(x))} = \frac{f(x)-f(y)}{g(x)-g(y)} \approx \frac{f(x)}{g(x)}$$

Dk: (ii) $\lim\limits_{x \to a_+} \frac{f'(x)}{g'(x)} = A$




$A \in \mathbb{R}$ (případ $A = \pm\infty$ je ideově podobný)

Nechť $\varepsilon > 0$. Existuje $\delta_1 > 0$, že

$$\forall x \in P_+(a, \delta_1): \quad \left| \frac{f'(x)}{g'(x)} - A \right| < \varepsilon \qquad (*)$$

Zafixujme si $y \in P_+(a, \delta_1)$.

Zvolme $0 < \delta_2 < \delta_1$, že $\forall x \in P_+(a, \delta_2)$: $\frac{1}{|g(x)|}\left(|f(y)| + |g(y)|(|A| + \varepsilon)\right) < \varepsilon$.

Zvolme libovolné $x \in P_+(a, \delta_2) \cap (a, y)$. Na intervalu $[x, y]$ jsou splněny předpoklady V4.9, tj. $\exists\, \xi \in (x, y)$

$$\frac{f(y) - f(x)}{g(y) - g(x)} = \frac{f'(\xi)}{g'(\xi)} \qquad / \cdot (g(y) - g(x))$$

$$f(y) - f(x) = \frac{f'(\xi)}{g'(\xi)} \cdot g(y) - \frac{f'(\xi)}{g'(\xi)} \cdot g(x) \qquad / \cdot \frac{1}{g(x)}$$

$$\underline{\frac{f(y)}{g(x)}} - \underline{\frac{f(x)}{g(x)}} = \underline{\frac{f'(\xi)}{g'(\xi)} \cdot \frac{g(y)}{g(x)}} - \underline{\frac{f'(\xi)}{g'(\xi)}}$$

Tedy máme $\forall \varepsilon > 0\ \exists \delta_2 > 0\ \forall x \in P_+(a, \delta_2)$

$$\left|\frac{f(x)}{g(x)} - A\right| \leq \left|\frac{f(x)}{g(x)} - \frac{f'(\xi)}{g'(\xi)}\right| + \left|\frac{f'(\xi)}{g'(\xi)} - A\right| <$$

$$< \varepsilon + \varepsilon$$

$(*)$

$\square$

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$$\left|\frac{f(x)}{g(x)} - \frac{f'(\xi)}{g'(\xi)}\right| = \left|\frac{f(y)}{g(x)} - \frac{f'(\xi)}{g'(\xi)} \cdot \frac{g(y)}{g(x)}\right| \leq$$

$$\leq \frac{|f(y)|}{|g(x)|} + \left|\frac{f'(\xi)}{g'(\xi)}\right| \cdot \frac{|g(y)|}{|g(x)|} \overset{(*)}{\leq} \frac{|f(y)|}{|g(x)|} + (|A| + \varepsilon) \cdot \frac{|g(y)|}{|g(x)|}$$

$$\leq \varepsilon$$

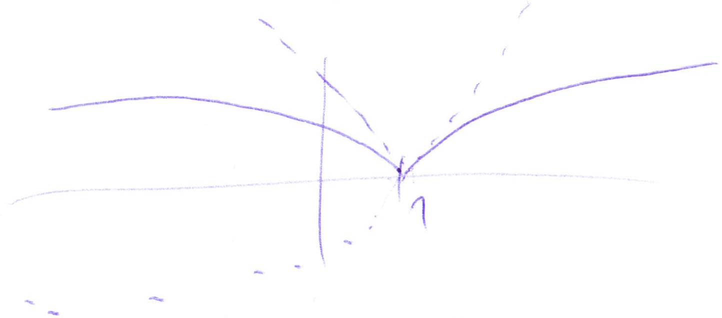
**Věta L 4.17** (derivace a limita derivace)

Nechť je funkce $f$ spojitá zprava v $a$ a nechť existuje $\lim_{x\to a+} f'(x) = A \in \mathbb{R}^*$. Pak $f'_+(a) = A$.

Dk: Z definice derivace

$$f'_+(a) = \lim_{x\to a+} \frac{f(x)-f(a)}{x-a} \overset{\frac{0}{0}}{\underset{l'H}{=}} \lim_{x\to a+} \frac{f'(x)}{1} = A$$

$f(x)-f(a) \xrightarrow{x\to a+} 0$, neboť $f$ je spojitá v $a$ zprava □



**Příklad:** $f(x) = |\arctan(x-1)|$ $f'_+(1)$, $f'_-(1)$

$$\cancel{f'_+(1) = \lim_{h\to 0} \frac{f(1+h)-f(1)}{h}}$$

$$f(x) = \begin{cases} \arctan(x-1) & \text{na } [1,\infty) \\ -\arctan(x-1) & \text{na } (-\infty,1] \end{cases}$$

$$f'(x) = \begin{cases} \frac{1}{1+(x-1)^2}\cdot 1 & \text{na } (1,\infty), \\ -\frac{1}{1+(x-1)^2}\cdot 1 & \text{na } (-\infty,1) \end{cases}$$

$f$ spojitá v 1

$$f'_+(1) = \lim_{x\to 1+} f'(x) = \lim_{x\to 1+} \frac{1}{1+(x-1)^2} = 1$$

$$f'_-(1) \overset{f \text{ spojitá}}{=} \lim_{x\to 1-} f'(x) = \lim_{x\to 1-} \frac{-1}{1+(x-1)^2} = -1$$

# 4.2. Konvexní a konkávní funkce

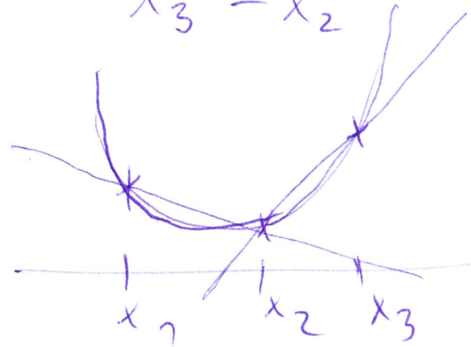
**Def** Nechť $n \in \mathbb{N}$, $a \in \mathbb{R}$ a nechť $f$ má vlastní $n$-tou derivaci na okolí bodu $a$. Pak $(n+1)$-ní derivací v bodě $a$ budeme rozumět $f^{(n+1)}(a) = \lim_{h \to 0} \frac{f^{(n)}(a+h) - f^{(n)}(a)}{h}$.

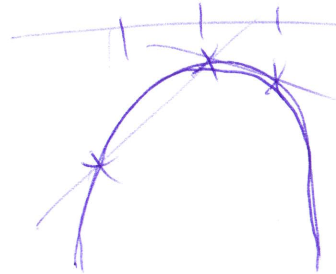
**Příklad:** $f^{(2)}(x)$, $f''(x)$, $f^{(3)}$, $f'''$

$$(x^2 \cdot e^x)'' = (2x \cdot e^x + x^2 \cdot e^x)' = 2 \cdot e^x + 2x \cdot e^x + 2x \cdot e^x + x^2 \cdot e^x$$

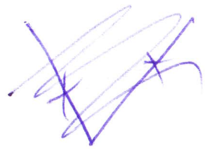
**Definice** Funkce $f$ na intervalu $I$ nazveme konvexní (konkávní), jestliže $\forall x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3 \Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \le \frac{f(x_3) - f(x_2)}{x_3 - x_2}$. $(\ge)$

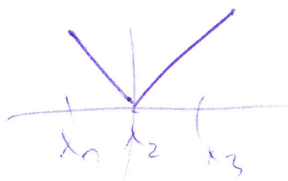
Funkci nazveme ryze konvexní (ryze konkávní), jsou-li příslušné nerovnosti ostré.



**Příklad:** Je $|x|$ konvexní na $[-1, 1]$

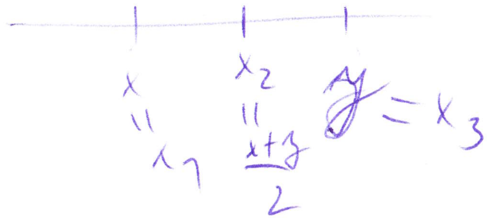




**Poznámka:** Ekvivalentně lze definovat, že funkce $f$ je na $I$ konvexní, pokud

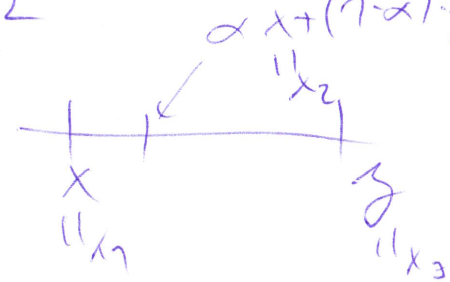
$$\forall x, y \in I,\ x < y\ \forall \alpha \in (0,1)$$
$$f(\alpha \cdot x + (1-\alpha) \cdot y) \le \alpha \cdot f(x) + (1-\alpha) \cdot f(y)$$

IDEA PROČ: $\alpha = \frac{1}{2}$ $\quad f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$



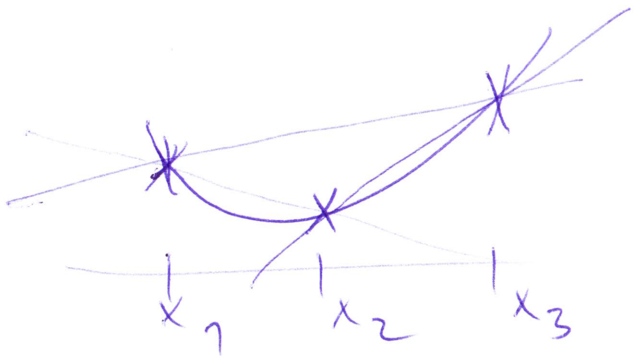

definice konvexity $x_1 = x,\ x_3 = y,\ x_2 = \frac{x+y}{2}$

$$\frac{f\left(\frac{x+y}{2}\right) - f(x)}{\cancel{\frac{y-x}{2}}} \le \frac{f(y) - f\left(\frac{x+y}{2}\right)}{\cancel{\frac{y-x}{2}}} \iff f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2}$$




**Lemma** Nechť je funkce $f$ na intervalu $I$ konvexní, pak

$$\forall x_1, x_2, x_3 \in I,\ x_1 < x_2 < x_3 \implies \frac{f(x_2)-f(x_1)}{x_2 - x_1} \le \frac{f(x_3)-f(x_1)}{x_3 - x_1} \le \frac{f(x_3)-f(x_2)}{x_3 - x_2}$$



**Lemma** Nechť je funkce $f$ na intervalu $I$ ryze konvexní, pak $\forall x_1, x_2, x_3 \in I$, $x_1 < x_2 < x_3$:

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} < \frac{f(x_3)-f(x_1)}{x_3-x_1} < \frac{f(x_3)-f(x_2)}{x_3-x_2}.$$

**Dk Lemmatu** $x_1 < x_2 < x_3$. Víme $\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2}$

Chceme $\frac{f(x_2)-f(x_1)}{x_2-x_1} \le \frac{f(x_3)-f(x_1)}{x_3-x_1}$.

$$\frac{1}{x_3-x_1}\left(f(x_3)-f(x_1)\right) = \frac{1}{x_3-x_1}\left(f(x_3)-f(x_2)+f(x_2)-f(x_1)\right)$$

$$\ge \frac{1}{x_3-x_1}\left(\frac{f(x_2)-f(x_1)}{x_2-x_1}\cdot(x_3-x_2) + f(x_2)-f(x_1)\right)$$

$$= \left(f(x_2)-f(x_1)\right)\cdot\frac{1}{x_3-x_1}\left(\frac{x_3-\cancel{x_2}+\cancel{x_2}-x_1}{x_2-x_1}\right)$$

$$= \frac{f(x_2)-f(x_1)}{x_2-x_1}$$

Druhá nerovnost $\frac{f(x_3)-f(x_1)}{x_3-x_1} \le \frac{f(x_3)-f(x_2)}{x_3-x_2}$ se ukáže analogicky $\square$