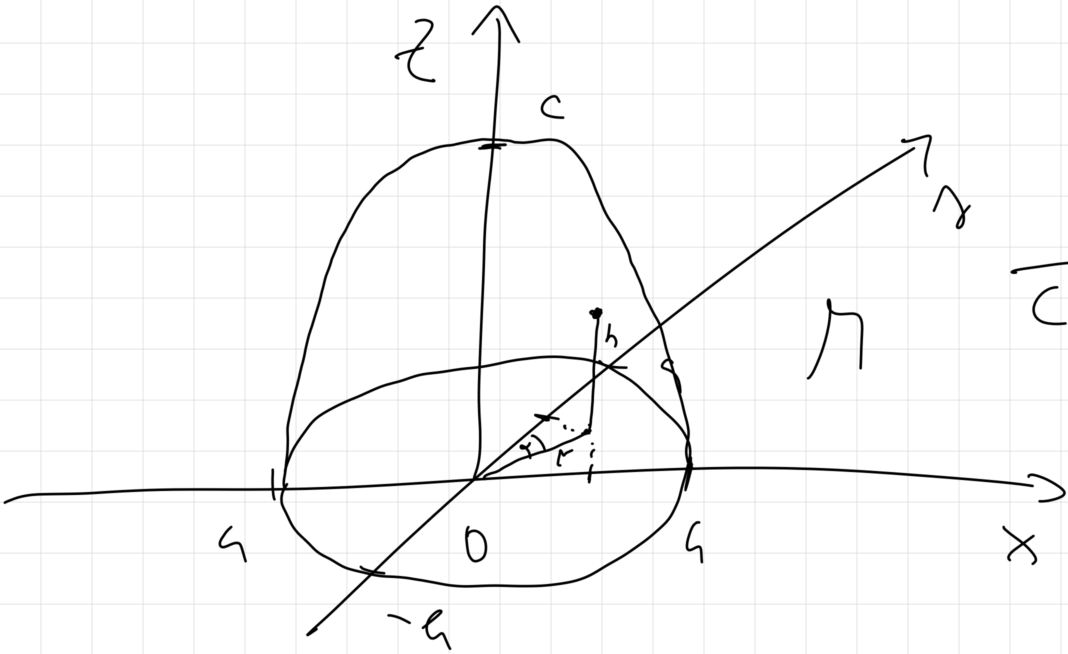
1) Určete $\lambda^3(M)$, kde $M = \{(x,y,z) \in \mathbb{R}^3;$

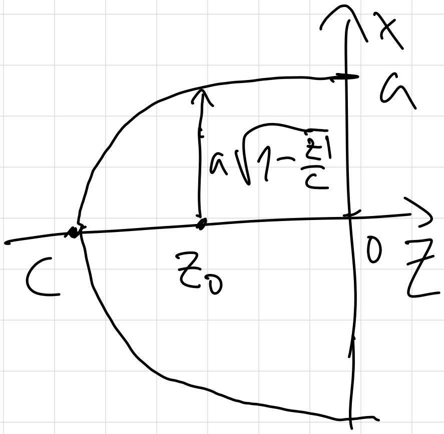
$c(x^2+y^2) + a^2 z \leq a^2 c$, $z \geq 0\}$

$$\boxed{\begin{aligned} \frac{x^2+y^2}{a^2} &\leq 1 - \frac{z}{c} \\ \sqrt{x^2+y^2} &\leq a\sqrt{1-\frac{z}{c}} \end{aligned}}$$

$[z \in [0,c]]$



z
c
y
h
a
r
a
0
a
x
-a



x
a
$a\sqrt{1-\frac{|z|}{c}}$
c
z_0
0
z

A je uz. množina $\Rightarrow$ A je měřitelná.

$$\lambda^3(A) = \int_A 1 \, d\lambda^3(x,y,z) \overset{(*)}{=} \int r \chi_A(\varphi(r,\alpha,h))$$

Substituce $(0,\infty)\times(-\pi,\pi)\times\mathbb{R}$

Fubini $r\chi_A \geq 0$

$$d\lambda^3(r,\alpha,h) = \int_{-\pi}^{\pi} \int_0^{c} \int_0^{a\sqrt{1-h/c}} r \; dr\, dh\, d\alpha$$

$$(*) : \varphi : (0,\infty) \times (-\pi, \pi) \times \mathbb{R} \to \mathbb{R}^3$$

$$(r, \alpha, h) \longmapsto (r \sin(\alpha), r\cos(\alpha), z)$$

$$\begin{pmatrix} x = r\sin(\alpha) \\ y = r\cos(\alpha) \\ z = h \end{pmatrix}$$

$$J\varphi(r, \alpha, h) = \begin{vmatrix} \sin(\alpha) & r\cos(\alpha) & 0 \\ \cos(\alpha) & r(-\sin(\alpha)) & 0 \\ 0 & 0 & 1 \end{vmatrix} = -r$$

φ je C^1, $J\varphi \neq 0$, φ prosté

$$\lambda^3(\mathbb{R}^3 \setminus \varphi(D_4)) = 0$$

$$= \int_{-\pi}^{\pi} \int_{0}^{c} \left[\frac{r^2}{2}\right]_{0}^{a\sqrt{1-\frac{h}{c}}} \, dh \, d\alpha =$$

$$= \int_{-\pi}^{\pi} \int_{0}^{c} \frac{a^2}{2}\left(1-\frac{h}{c}\right) dh \, d\alpha =$$

$$= \int_{-\pi}^{\pi} \frac{a^2}{2}\left[h - \frac{h^2}{2c}\right]_{0}^{c} d\alpha = \frac{a^2 c}{2}\pi$$

2) Určete $\lambda^3(A)$ kde $A = \{(x,y,z) \in \mathbb{R}^3 ;$

$$\frac{y^2}{2} + \frac{z^2}{3} \leq \frac{1}{x^2+2x+3} \}$$

$$\boxed{\frac{y^2}{2} + \frac{z^2}{3} \leq \frac{1}{(x+1)^2+2}}$$

$$\left[P_0 \ (*): \quad r^2 \leq \frac{1}{h^2+1} \iff r \leq \sqrt{\frac{1}{h^2+1}} \right]$$

$$\frac{y^2}{2} + \frac{z^2}{3} = r^2 \qquad \begin{aligned} y &= r\sin(\alpha)\sqrt{2} \\ z &= r\cos(\alpha)\sqrt{3} \\ x &= h-1 \end{aligned} \qquad (*)$$

$$\varphi : (0,\infty) \times (-\pi,\pi) \times \mathbb{R} \to \mathbb{R}^3, \quad \lambda^3(\mathbb{R}^3 \setminus \varphi(D_\varphi)) = 0$$

$$(r,\alpha,h) \longmapsto (h-1,\ r\sin(\alpha)\sqrt{2},\ r\cos(\alpha)\sqrt{3})$$

$$|J\varphi(r,\alpha,h)| = \left\| \begin{matrix} 0 & 0 & 1 \\ \sin(\alpha)\sqrt{2} & r\cos(\alpha)\sqrt{2} & 0 \\ \cos(\alpha)\sqrt{3} & r(-\sin(\alpha))\sqrt{3} & 0 \end{matrix} \right\| = r\sqrt{6}$$

φ prostá

$J\varphi \neq 0$

φ je C^1

$$\lambda^3(\Omega) = \int_{\mathbb{R}^3} \chi_\Omega(x,y,z)\, d\lambda^3(x,y,z) \overset{\text{Substituce } (*)}{=}$$

$$= \int_{(0,\infty)\times(-\pi,\pi)\times\mathbb{R}} \chi_\Omega(\varphi(r,\alpha,h)) \cdot r\sqrt{6}\, d\lambda^3(r,\alpha,h) =$$

$\chi_\Omega \cdot r \cdot \sqrt{6} \geq 0$, Fubini

$$= \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{0}^{\sqrt{\frac{1}{h^2+2}}} r\sqrt{6}\, dr\, dh\, d\alpha =$$

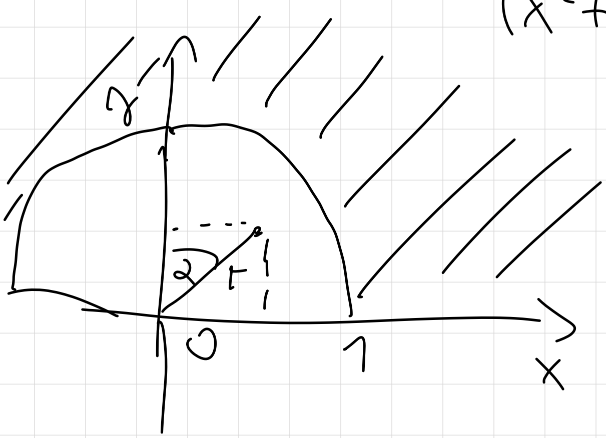
$$= \sqrt{6} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \frac{1}{2} \frac{1}{h^2+2} \, dh \, d\lambda =$$

$$= \pi\sqrt{6} \int_{-\infty}^{\infty} \frac{1}{2} \cdot \frac{1}{1+\left(\frac{h}{\sqrt{2}}\right)^2} \, dh =$$

$$\frac{\pi\sqrt{6}}{2} \left[\operatorname{arctg}\left(\frac{h}{\sqrt{2}}\right) \cdot \sqrt{2} \right]_{-\infty}^{\infty} = \pi^2 \sqrt{3}.$$

3) Spočtěte $\int_{\Omega} f(x,y)\, d\lambda^2(x,y)$ kde

$f(x,y) = \dfrac{\ln(x^2+y^2)}{(x^2+y^2)^2}$, $\Omega = \{(x,y) \in \mathbb{R}^2;\ y \geq 0,\ x^2+y^2 \geq 1\}$



$x = r\sin(\alpha)$

$y = r\cos(\alpha)$

$(*)$

$\varphi : (0,\infty)\times(-\pi,\pi) \to \mathbb{R}^2$

$$J\varphi = \begin{vmatrix} \sin(\alpha) & r\cos(\alpha) \\ \cos(\alpha) & -r\sin(\alpha) \end{vmatrix} = -r$$

$$\lambda^2(\mathbb{R}^2 \setminus \varphi(D_\varphi)) = 0$$

φ je prosté a C^1, $J\varphi \neq 0$

$$\varphi^{-1}(\Lambda) = [1,\infty)\times\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$$

$$\int_{M} f(x,y)\, d\lambda^2(x,y) \overset{(*)}{=} \int_{[1,\infty)\times(-\frac{\pi}{2},\frac{\pi}{2})} \frac{\log(r^2)}{r^4}\, r\, d\lambda^2(r,\alpha)$$

(*) Substituce

$$\overset{(\Delta)}{=} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{\infty} \frac{2\log(r)}{r^3}\, dr\, d\alpha = 2\pi \int_{1}^{\infty} \frac{2\log(r)}{r^3}\, dr \dots$$

(Δ) integrand je na množině nezáporný, Fubini