Cv. 15 – 24.11.

Obecná mocnina: $x^y := \exp(y \cdot \lg x)$, $x, y \in \mathbb{R}$, $x > 0$

$(x \quad x^n;\ n \in \mathbb{N};$

$x^{\frac{1}{n}};\ n \in \mathbb{N})$

1) $$\lim_{x \to 0} (1+4x)^{\frac{1}{3x}} = \lim_{x \to 0} \exp\left(\frac{1}{3x} \lg(1+4x)\right) =: L$$

$$\lim_{x \to 0} \frac{1}{3x} \lg(1+4x) = \lim_{x \to 0} \underbrace{\frac{\lg(1+4x)}{4x}} \cdot \frac{4\cancel{x}}{3\cancel{x}} = \frac{4}{3}$$

vnější: $\frac{\lg(1+y)}{y} \xrightarrow[y \to A = 0]{} 1$

vnitř? $4x \xrightarrow[x \to 0]{} 0 =: A$

$\underbrace{\text{(P) } 4x \text{ je prostá.}}$

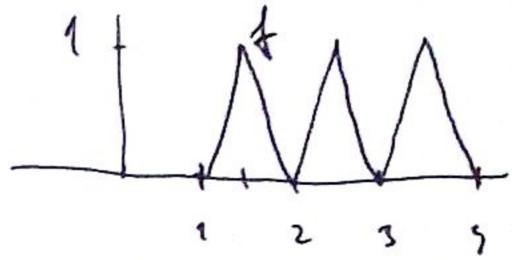
$\xrightarrow[x \to 0]{} 1$

$L = e^{\frac{4}{3}}$, protože exp je spojitá v $\frac{4}{3}$.

3) $$\lim_{n\to+\infty}\left(\frac{n^2+1}{n^2-1}\right)^{\sqrt{n^3+3n^2}} = LP$$

Nejdříve spočteme limitu pro:

$$\lim_{x\to+\infty}\left(\frac{x^2+1}{x^2-1}\right)^{\sqrt{x^3+3x^2}} =$$



$f(n)=0 \xrightarrow[n\to+\infty]{} 0$

ale $f(x) \underset{x\to+\infty}{\not\to} 0$

$$\lim_{x\to+\infty}\exp\left(\sqrt{x^3+3x^2}\,\lg\left(\frac{x^2+1}{x^2-1}\right)\right) = LF$$

Spočtu: $$\lim_{x\to+\infty}\sqrt{x^3+3x^2}\,\lg\left(\frac{x^2+1}{x^2-1}\right) \overset{AL}{=} 0$$

$$\lim_{x\to+\infty}\underbrace{\frac{\lg\left(1+\left(\frac{x^2+1}{x^2-1}-1\right)\right)}{\frac{x^2+1}{x^2-1}-1}}\cdot\left(\frac{x^2+1}{x^2-1}-1\right)\sqrt{x^3+3x^2}$$

vnější: $$\frac{\lg(1+y)}{y}\xrightarrow[y\to A=0]{} 1$$

vnitřní: $$\frac{x^2+1}{x^2-1}-1=\frac{2}{x^2-1}\xrightarrow{x\to+\infty} 0=:A$$

$$\underbrace{\text{(P)}\ \frac{2}{x^2-1}\neq 0 \text{ na svém df.}}_{\xrightarrow{x\to+\infty} 1}$$

$$\left(\frac{x^2+1}{x^2-1}-1\right)\sqrt{x^3+3x^2} = \frac{2}{x^2-1}\sqrt{x^3}\sqrt{1+3\frac{1}{x}}$$

$$= \underbrace{\frac{\sqrt{x^3}}{x^2}}_{\substack{\longrightarrow 0 \\ x\to+\infty}} \underbrace{\frac{2\sqrt{1+\frac{3}{x}}}{1-\frac{1}{x^2}}}_{\substack{\longrightarrow 2 \\ x\to+\infty}} \xrightarrow{x\to+\infty} 0.$$

$LF = \underset{\overset{\parallel}{1}}{e^0}$, protože exp je spojitá v 0.

$LP = 1$ podle Heineovy věty, $(\{a_n\},\ a_n = n)$.

Jinak:

$$LP = \lim_{n\to+\infty}\left(\frac{1+\frac{1}{n^2}}{1-\frac{1}{n^2}}\right)^{\sqrt{\frac{1+3/n}{\left(\frac{1}{n}\right)^3}}} = 1$$

Heine: Počítáme limitu pro:

$$\lim_{t\to 0+}\left(\frac{1+t^2}{1-t^2}\right)^{\sqrt{\frac{1+3t}{t^3}}} \ldots\ldots = e = 1$$

Použijme Heine: $\{a_n\} = \{\frac{1}{n}\}$

K5)

$$\lim_{x\to+\infty} \underbrace{\frac{x^3}{1-x}}_{\xrightarrow[x\to+\infty]{}-\infty} \underbrace{\lg\left(\underbrace{\frac{3x^2-x+1}{2x^2+x+1}}_{\xrightarrow[x\to+\infty]{}\frac{3}{2}}\right)}_{\xrightarrow[x\to+\infty]{}\lg\frac{3}{2}>0} = -\infty$$

(P) ne Vo LSF

K6)

$$\frac{1}{x^2}\lg\left(\frac{1+x2^x}{1+x3^x}\right) = \lg\left(1+\overbrace{\frac{1+x2^x-1-x3^x}{1+x3^x}}^{\frac{x(2^x-3^x)}{1+x3^x}}\right)\frac{1}{x^2}$$

$$= \frac{\log\left(1+\frac{x(2^x-3^x)}{1+x3^x}\right)}{\frac{x(2^x-3^x)}{1+x3^x}}\cdot\frac{2^x-3^x}{1+x3^x}\,\frac{1}{x}$$

16. cvičení murma 101 – 25.11.20

$n \in \mathbb{N}$

2. $\lim_{y \to 0} \frac{e^y - 1}{y} = 1$

$$\lim_{x \to 0} \frac{\sqrt[n]{1+x} - 1}{x} = \lim_{x \to 0} \frac{\exp\left(\frac{1}{n} \lg(1+x)\right) - 1}{x} =$$

$$= \lim_{x \to 0} \underbrace{\frac{\exp\left(\frac{1}{n} \lg(1+x)\right) - 1}{\frac{1}{n} \lg(1+x)}}_{\xrightarrow[x \to 0]{} 1} \cdot \underbrace{\frac{\frac{1}{n} \lg(1+x)}{x}}_{\xrightarrow[x \to 0]{} 1} = \frac{1}{n}$$

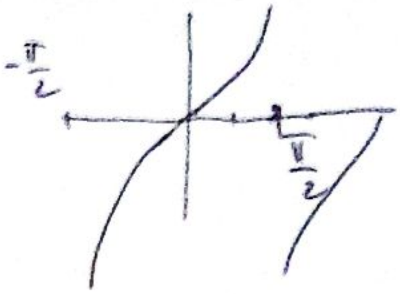
VoLSF: vnější: $\frac{e^y - 1}{y} \xrightarrow[y \to A(=0)]{} 1$

vnitřní: $\frac{1}{n} \lg(1+x) \xrightarrow[x \to 0]{} 0 =: A$

(P) platí protože $\frac{1}{n} \lg(1+x)$ je rostoucí

DÚ: vyřešit pomocí $A^n - B^n = (A-B)(A^{n-1} + \dots + B^{n-1})$

6)



$$\lim_{x\to\frac{\pi}{4}} (\operatorname{tg} x)^{\operatorname{tg} 2x} =$$

$$\lim_{x\to\frac{\pi}{4}} \exp\left(\operatorname{tg} 2x \lg \operatorname{tg} x\right) =: L$$

$$\lim_{x\to\frac{\pi}{4}} \operatorname{tg} 2x \lg(\operatorname{tg} x) = \lim_{x\to\frac{\pi}{4}} \underbrace{\frac{\lg(1+[\operatorname{tg} x - 1])}{\operatorname{tg} x - 1}}_{\xrightarrow[x\to\frac{\pi}{4}]{} 1} \cdot$$

$$\cdot (\operatorname{tg} x - 1)\cdot \operatorname{tg}(2x) \overset{AL}{=}$$

vnější: $\frac{\lg(1+y)}{y} \xrightarrow[y\to A(=0)]{} 1$

vnitřní: $\operatorname{tg} x - 1 \xrightarrow[x\to\frac{\pi}{4}]{} 0 =: A$

(P) $\operatorname{tg} x - 1$ je nenulová na $P(\frac{\pi}{4}, \frac{\pi}{4})$

$$L = \lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\cos x} \cdot \frac{\sin 2x}{\cos 2x} \overset{AL}{=}$$

$$\left(\lim_{x\to\frac{\pi}{4}} \frac{\sin x - \cos x}{\cos 2x}\right)\cdot \frac{2}{\sqrt{2}} = \frac{2}{\sqrt{2}} \lim_{x\to\frac{\pi}{4}} \frac{\sin(x - \frac{\pi}{4} + \frac{\pi}{4}) - \cos(x - \frac{\pi}{4} + \frac{\pi}{4})}{\cos[2(x - \frac{\pi}{4} + \frac{\pi}{4})]}$$

$$\left[\begin{aligned} &\sin(a \pm b) = \sin a \cos b \pm \sin b \cos a \\ &\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b \end{aligned}\right.$$

$$= \frac{2}{\sqrt{2}} \lim_{x\to\frac{\pi}{4}} \frac{\sin(x-\frac{\pi}{4})\frac{\sqrt{2}}{2} + \cancel{\frac{\sqrt{2}}{2}\cos(x-\frac{\pi}{4})} - \left(\cancel{\cos(x-\frac{\pi}{4})\frac{\sqrt{2}}{2}} - \sin(x-\frac{\pi}{4})\frac{\sqrt{2}}{2}\right)}{\cos(2(x-\frac{\pi}{4}))\cos\frac{\pi}{2} - \sin 2(x-\frac{\pi}{4})\sin(2\frac{\pi}{4})}$$

$$= \frac{2}{\sqrt{2}} \lim_{x \to \frac{\pi}{4}} \underbrace{\frac{\sqrt{2} \sin\left(x - \frac{\pi}{4}\right)}{x - \frac{\pi}{4}}}_{\xrightarrow[x \to \frac{\pi}{4}]{} 1} \cdot \underbrace{\frac{2\left(x - \frac{\pi}{4}\right)}{-\sin 2\left(x - \frac{\pi}{4}\right)}}_{\xrightarrow[x \to \frac{\pi}{4}]{} 1} \frac{\cancel{x - \frac{\pi}{4}}}{2\cancel{\left(x - \frac{\pi}{4}\right)}}$$

$$= \frac{\cancel{2}}{\cancel{\sqrt{2}}} \frac{\cancel{\sqrt{2}}}{-1} \cdot \frac{1}{\cancel{2}} = -1$$

$L = e^{-1} = \frac{1}{e}$, protože exp je v -1 spojitá (VoLSF).

$$\frac{\sin x - \cos x}{\cos 2x} = \frac{\sin x - \cos x}{\cos^2 x - \sin^2 x} = \frac{\cancel{\sin x - \cos x}^{-1}}{\cancel{(\cos x - \sin x)}(\cos x + \sin x)}$$

$$= \frac{-1}{\cos x + \sin x} \longrightarrow \frac{-1}{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}} = -\frac{1}{\sqrt{2}} \checkmark$$

$$\lim_{x\to 0} \frac{\operatorname{tg} x - \sin x}{x^3} = \lim_{x\to 0} \frac{\frac{\sin x}{\cos x} - \sin x}{x^3} =$$

$$= \lim_{x\to 0} \frac{\sin x}{x} \cdot \frac{\left(\frac{1-\cos x}{\cos x}\right)}{x^2} \overset{AL}{=} \frac{1}{2}$$

$$\frac{\sin x}{x} \xrightarrow[x\to 0]{} 1 \qquad \frac{1-\cos x}{\cos x \, x^2} \xrightarrow[x\to 0]{} \frac{1}{2}$$

K7:

$$\frac{\frac{e^x - 1}{x} + \frac{1 - 2\sin\left(\frac{\pi}{6} + x\right)}{x}}{\frac{\operatorname{tg} x}{x}}$$