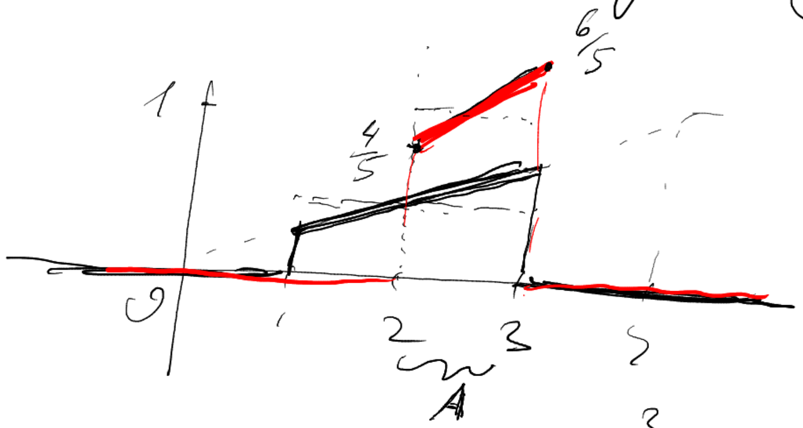
7. cvičení

(1)

X má hustotu $f_X = \begin{cases} x/4 & x \in (1,3) \\ 0 & \text{jinak} \end{cases}$



$$\mathbb{E}X = \int_{-\infty}^{\infty} x \cdot f(x) = \int_1^3 x \cdot \frac{x}{4} = \left[\frac{x^3}{3 \cdot 4}\right]_1^3 = \frac{27-1}{12} = \frac{13}{6}$$

$$P(A) = \int_2^{\infty} f(x) = \int_2^3 \frac{x}{4} = \left[\frac{x^2}{8}\right]_2^3 = \frac{5}{8}$$

(kontrola: $\int_{-\infty}^{\infty} f(x) = \int_1^3 \frac{x}{4} = \left[\frac{x^2}{8}\right]_1^3 = \frac{9-1}{8}$ ✓)

$$P[X \in S] = \int_S f_X(x)$$

$S = [2, \infty)$

$$f_{X|A} = \begin{cases} 0 & \text{mimo } [2,3] \\ \frac{f_X}{P(A)} = \frac{2}{5}x & x \in [2,3] \end{cases}$$

$$\mathbb{E}[X|A] = \int_2^3 x \cdot \underline{f_{X|A}}(x)$$

$$= \int_2^3 \frac{2}{5} x^2 = \left[\frac{2}{15} x^3\right]_2^3 = \frac{2}{15} \cdot (27-8)$$

$$= \underline{\underline{\frac{2 \cdot 19}{15}}}$$

~~jiný postup~~

~~$\underset{\sim}{\mathbb{E}}(X) = \mathbb{E}[X|A]\, P(A) + \underline{\mathbb{E}(X|A^c)}\, P(A^c)$~~

~~$\frac{13}{6} = 9 \cdot \frac{5}{8} + \underbrace{(9 -}_{}\quad \frac{3}{8}$~~

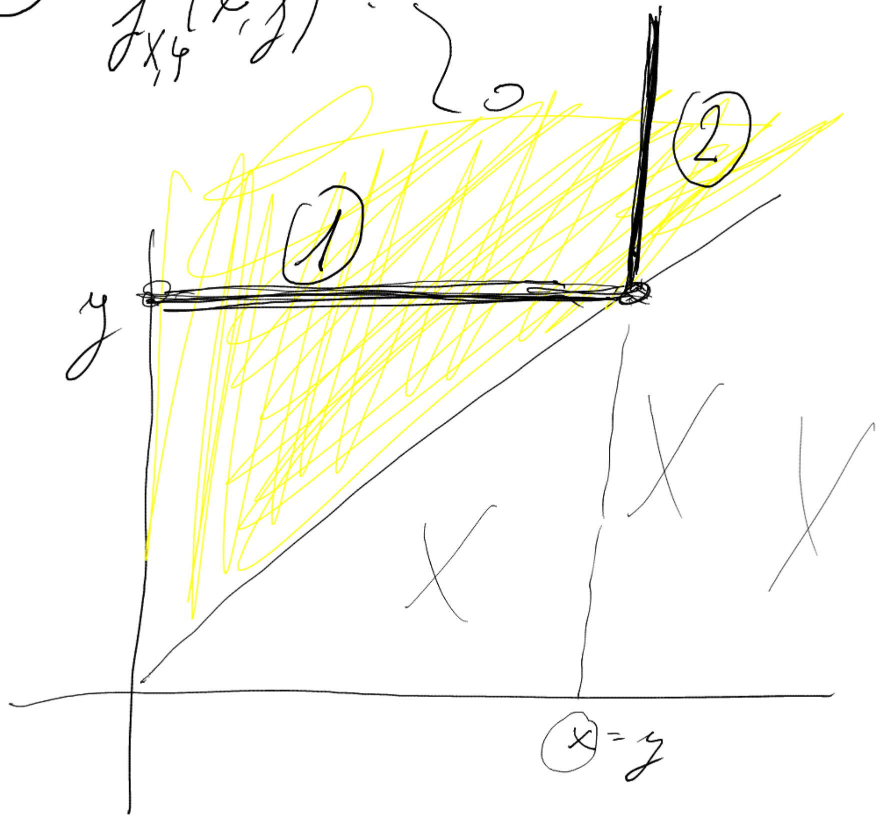
b) $Y = X^2$

$$\mathbb{E} Y = \int_{-\infty}^{\infty} x^2 f(x) = \int_1^3 x^2 \cdot \frac{x}{4} = \left[\frac{x^4}{16}\right]_1^3$$

$$= \frac{80}{16} = 5$$

var $Y = \ldots$

② $f_{X,Y}(x,y) = \begin{cases} e^{-y} & 0 < x < y < \infty \\ 0 \end{cases}$



$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \boxed{\frac{1}{y}}$$

1

$$\underline{f_Y(y)} = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,\underline{dx} = \int_0^y e^{-y}\,dx =$$

$$= \underline{y e^{-y}}$$

(kontrola:

$$\int_0^\infty y e^{-y} = [-y e^{-y}]_0^\infty - \int_0^\infty -e^{-y} = 1)$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$= \int_x^{\infty} e^{-y}\,dy = \left[-e^{-y}\right]_x^{\infty} = e^{-x}$$

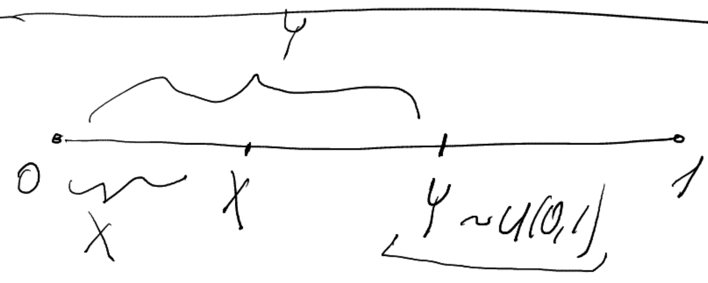
2

$$\left(\int_0^{\infty} e^{-x} = 1\right)$$

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{e^{-y}}{e^{-x}} = e^{x-y}$$

$$Y|X=x \sim x + Exp(1)$$

3



$$X \sim U(0,Y)$$

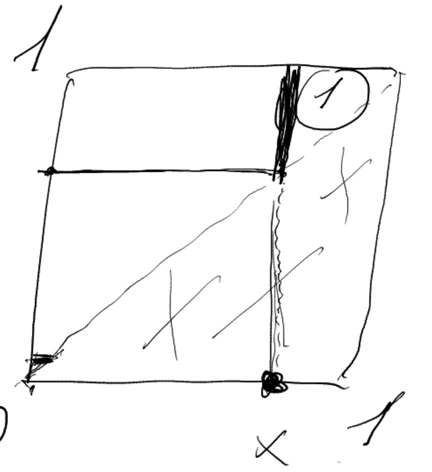
a

$$f_{X|Y}(x|y) = \begin{cases} \frac{1}{y} & x \in (0,y) \\ 0 & \text{jinak} \end{cases} \qquad y \in (0,1)$$

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)\cdot f_Y(y)$$

$$= \frac{1}{y}\cdot 1 \qquad 0 \le x \le y \le 1$$

(b) $f_X(x) = \int_0^1 f_{X,Y}(x,y)\,dy$



$= \int_{\cancel{x}}^1 \frac{1}{y}\,dy = [\ln y]_x^1 = -\ln x$ pro $x \in (0,1)$

1

$\left(\int_0^1 -\ln x = 1\right)$

(c)

$$E(X) = \int_0^1 x \cdot f_X(x)\,dx$$

$$= \int_0^1 -x \ln x = \underbrace{\left[-\frac{x^2}{2}\ln x\right]_0^1}_{0} + \underbrace{\int_0^1 \frac{x^2}{2}\frac{1}{x}}_{\frac{1}{4}} = \frac{1}{4}$$

(d) $X = Y \cdot \frac{X}{Y}$ $\qquad \frac{X}{Y} \sim U(0,1)$ pro lib. $Y = y$

$$E[X|Y=y] = y \cdot \frac{1}{2}$$

$$E[X] = \int_0^1 E(X|Y=y)\cdot \underbrace{f_Y(y)}_{1} = \int_0^1 \frac{y}{2} = \left[\frac{y^2}{4}\right]_0^1 = \frac{1}{4}$$

$\int_0^1\int_0^1 x\, f_{X,Y}(x,y)$

(7) $S = \sum_{k=0}^{30} \binom{100}{k}$

$X = \sum_{i=1}^{100} X_i$

$X_i = \begin{cases} 1 & \text{s } \tfrac{1}{2} \\ -1 & \text{s pstí } \tfrac{1}{2} \end{cases}$

$\mathbb{E} X_i = 1$

$\operatorname{var} X_i = \sigma_{X_i} = 1$

(a)

$$\boxed{\text{Laplace-Moivre}}$$

$$\frac{\binom{100}{0}}{2^{100}} = P(X = 100)$$

$$\frac{\binom{100}{k}}{2^{100}} = P(X = \underline{\underline{100 - 2k}})$$

$$\frac{S}{2^{100}} = P(X \geq 100 - 2 \cdot 30 = 40)$$

(b) $\operatorname{var} X = \sum_{i=1}^{100} \operatorname{var} X_i = 1 \cdot 100$

$\sigma_X = 10, \quad \mathbb{E} X = 0$

Podle CLV $\frac{X - \mu}{\sigma} = \frac{X}{10}$ přibl. $\sim N(0,1)$

$Z \sim N(0,1)$

$$P(X \geq 40) \doteq P(Z \geq 4) = 1 - \Phi(4)$$

$$= 3.16 \cdot 10^{-5}$$

$$S \doteq 2^{100} \cdot 10^{-5} \cdot 3.16$$

přesně: $\boxed{2^{100} \cdot 10^{-5} \cdot 3.9}$

8 $\frac{1}{2^{100}}\binom{100}{30} = P(X = 40) \cancel{=} \underbrace{P(Z = 4)}_{= 0}$

$$\parallel$$

$$P(39.5 \leq X \leq 40.5) \doteq P(3.95 \leq Z \leq 4.05)$$

$$= \Phi(4.05) - \Phi(3.95)$$

$$\underbrace{\binom{100}{30}}_{2.9 \cdot 10^{25}} \doteq \underbrace{2^{100} \cdot (\Phi(4.05) - \Phi(3.95))}_{1.7 \cdot 10^{25}}$$