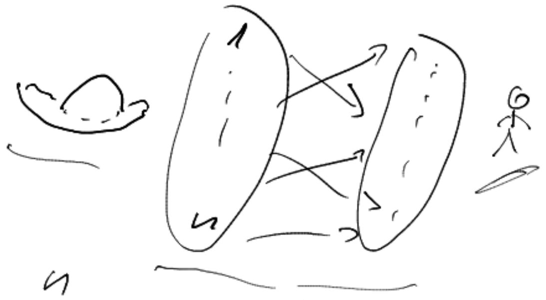
6. cvič. PSt

(1) $X = \sum_{i=1}^{n} X_i$ $\qquad X_i = \begin{cases} 0 \\ 1 \end{cases}$ pokud i-tý člověk dostane svůj klobouk



lin. střh.

indikátor. n. v.

$EX_i = P(\text{i-tý kl.} \to \text{i-tý čl.}) \cdot 1 = \frac{1}{n} = \frac{(n-1)!}{n!}$

$$EX = \sum_{i=1}^{n} EX_i = n \cdot \frac{1}{n} = 1$$

$$\operatorname{var}(X) = \sum_{i,j} \operatorname{cov}(X_i, X_j) = \sum_{i=1}^{n} \underbrace{\operatorname{var}(X_i)}_{p(1-p)} + \sum_{i \neq j} \operatorname{cov}(X_i, X_j)$$

Bern(p), $p = \frac{1}{n}$

$$EX_i^2 - (EX_i)^2 = EX_i - (EX_i)^2 = \frac{1}{n} - \frac{1}{n^2}$$

$X_i^2 = X_i$ (0, 1)

$$EX_iX_j = P(\text{i-tý \& j-tý dostanou svůj}) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

$i \to i$

$j \to j$

$n-2 \{$

$$EX_iX_j - (EX_i)(EX_j)$$

$$\frac{1}{n(n-1)} - \frac{1}{n} \cdot \frac{1}{n} = \frac{1}{n^2(n-1)} \neq 0$$

$$\operatorname{var} X = n \cdot \left(\frac{1}{n} - \frac{1}{n^2}\right) + n(n-1) \cdot \frac{1}{n^2(n-1)} = 1 - \frac{1}{n} + \frac{1}{n} = 1$$

$\sigma_X = 1$

(2) $X_1 \dots X_n$ ... $\mu, \sigma^2$ nezávislé!

$$S_n = \frac{X_1 + \dots + X_n}{n}$$

$EX_iX_j = EX_i \cdot EX_j \Rightarrow cov(X_i, X_j) = 0$

$i \neq j$

a) $\underline{ES_n} = \frac{1}{n}(EX_1 + \dots + EX_n) = \frac{1}{n}(\mu + \dots + \mu) = \underline{\underline{\mu}}$

$$\underline{var(S_n)} = \frac{1}{n^2} \cdot (\underbrace{var X_1 + \dots + var X_n}_{+0}) = \frac{1}{n^2}(\sigma^2 + \dots + \sigma^2) = \underline{\underline{\frac{\sigma^2}{n}}}$$

b) $$\underline{\underline{S_n}} = \frac{X_1 + \dots + X_{n-1} + X_n}{n} = \frac{(n-1) S_{n-1} + X_n}{n}$$

$$= \frac{n-1}{n} \cdot S_n + \frac{X_n}{n}$$

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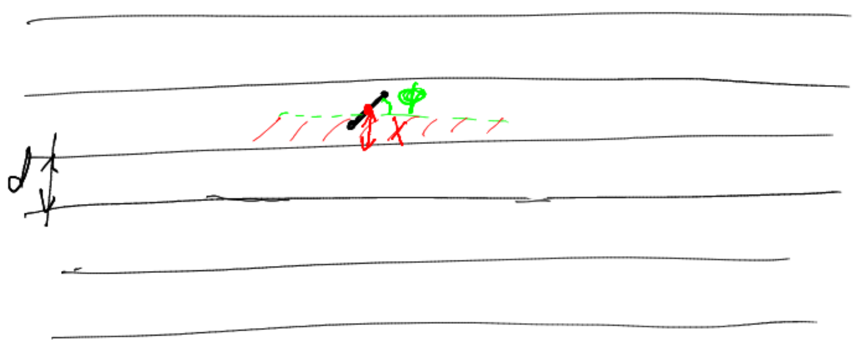
(3)



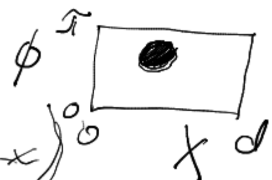

$\boxed{l < d}$

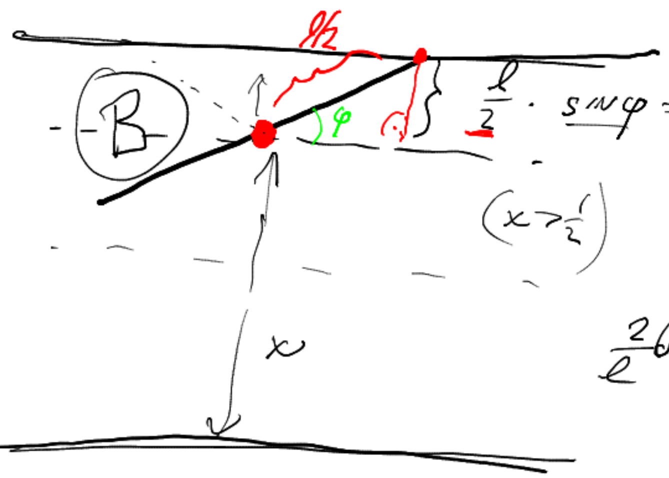
$X \sim U(0, d)$

$\Phi \sim U(0, \pi)$

$f_{X,\varphi} =$ const. na $[0,d] \times [0,\pi]$

$$\frac{1}{\pi d}$$



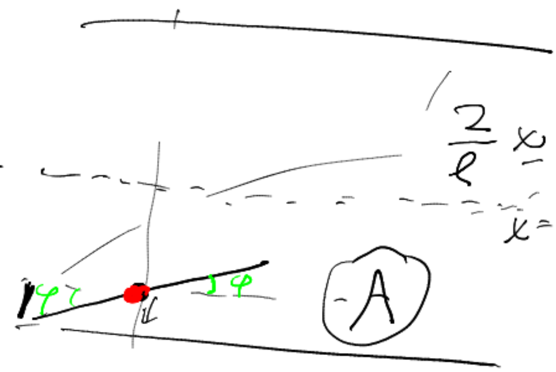




$\frac{l}{2} \cdot \sin\varphi = \underline{d - x}$

$(x > \frac{l}{2})$

~~$(\varphi = \arcsin \frac{2}{l}(d-x))$~~


j: protne čáru




$\frac{2}{l} x = \sin\varphi$

$x = \frac{l}{2} \sin\varphi$


$\frac{l}{2} \sin\varphi$

$$\iint\limits_{(x,\varphi)\in A} f(x,\varphi) = \int_0^{\pi}\int_0^{\frac{l}{2}\sin\varphi} \frac{1}{d\pi}\,dx\,d\varphi$$

$$= \int_0^{\pi} \frac{1}{\pi d}\cdot\frac{l}{2}\sin\varphi\,d\varphi$$

$$= \frac{l}{2\pi d}\Big[-\cos\varphi\Big]_0^{\pi} = \frac{\cancel{2}l}{\cancel{2}\pi d} = \frac{l}{\pi d}$$

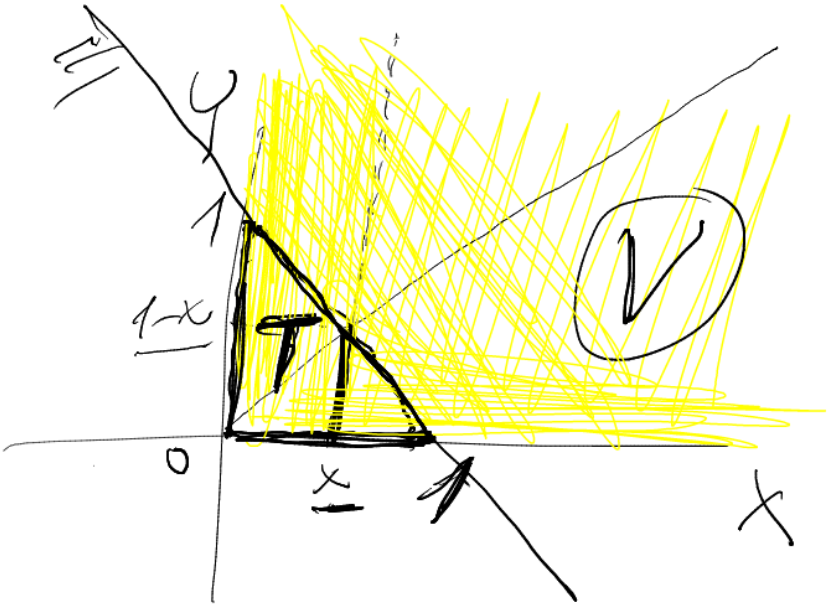
$$P(A) = \frac{l}{\pi d} = P(B)$$

$$\boxed{P(\text{jehla protne okraj prkena}) = \boxed{\frac{2l}{\pi d}}} = \mu$$

$$\boxed{\pi = \frac{2l}{\mu d}}$$

④ $X, Y$ --- $f_{X,Y}(x,y) = \underline{e^{-x-y}}$ $\quad x > 0, \; y > 0$

$= 0$ jinak




kontrola: $\iint f(x,y) = 1$ ?  $\quad \underline{e^{-x} \cdot e^{-y}}$

$$\int_0^\infty \int_0^\infty \underline{e^{-x-y}} = \int_0^\infty e^{-x} \cdot \int_0^\infty e^{-y} = \underline{1}$$

$$\left[-e^{-x}\right]_0^\infty = 0 - (-1) = \underline{1}$$

$P((X,Y) \in T)$

$\|$

$$\underline{P(X+Y \le 1)} = \iint_{(x,y)\in T} \underline{f(x,y)} = \iint_T \underline{e^{-x-y}}$$

$$= \underline{\int_0^1 \left( \int_0^{1-x} e^{-x-y} \, \underline{dy} \right) \underline{dx}}$$

$$= \int_0^1 \left[-e^{-x-y}\right]_0^{1-x} dx = \int_0^1 \left(-e^{-1} + e^{-x}\right)$$

$$= \int_0^1 -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} + \left(-e^{-1} + 1\right)$$

$$= \underline{\underline{1 - \frac{2}{e}}} \in [0,1]$$

$$P(X > Y) = \iint_V e^{-x-y} = \int_0^\infty \int_0^x e^{-x-y}\, dy\, dx$$

$$= \int_0^\infty \left[-e^{-x-y}\right]_0^x = \int_0^\infty \left(e^{-x} - e^{-2x}\right) dx = \left[-e^{-x}\right]_0^\infty - \frac{1}{2}\left[-e^{-2x}\right]_0^\infty$$

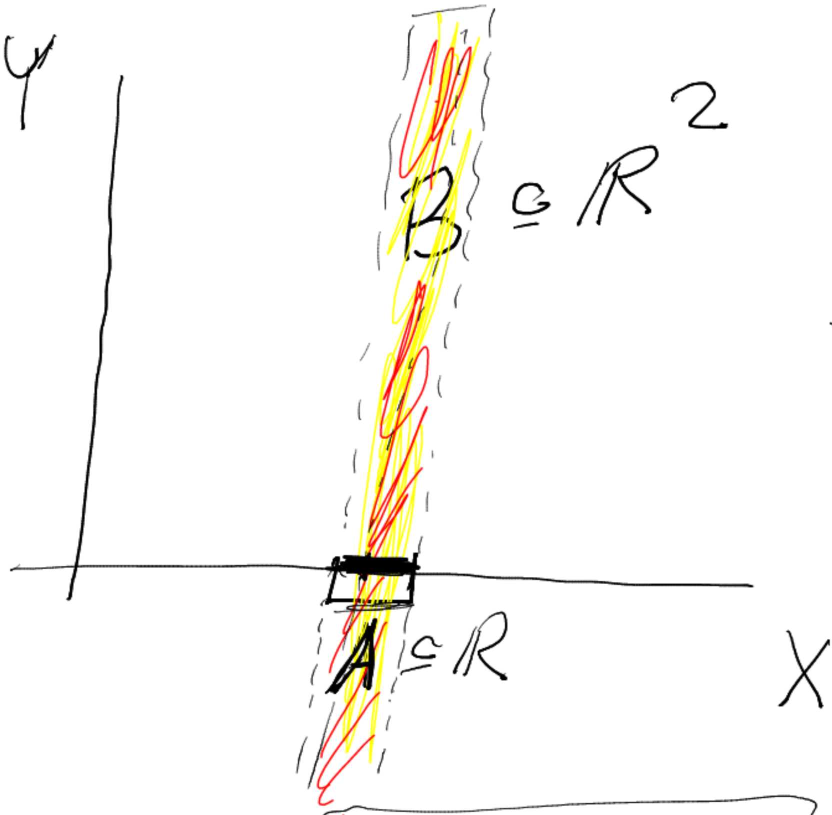
$$= 1 - \frac{1}{2} = \underline{\underline{\frac{1}{2}}}$$

c) $\boxed{f_X(\underline{x})} = \int_0^\infty e^{-x-y}\, dy = \left[-e^{-x-y}\right]_0^\infty$

$"\int_{-\infty}^{\infty} f_{X,Y}(x,y)\, dy"$ $= 0 + \underline{\underline{e^{-x}}}$

$f_Y(y) = \int \ldots = \underline{\underline{e^{-y}}}$

b) $\underline{\underline{f_{X,Y}(x,y)}} = \underline{\underline{f_X(x) \cdot f_Y(y)}}$ ... $X, Y$ nez.




$$B = A \times \mathbb{R}$$

$$P(X \in A) = \boxed{\int_A f_X(x)\, dx}$$

$$\|$$

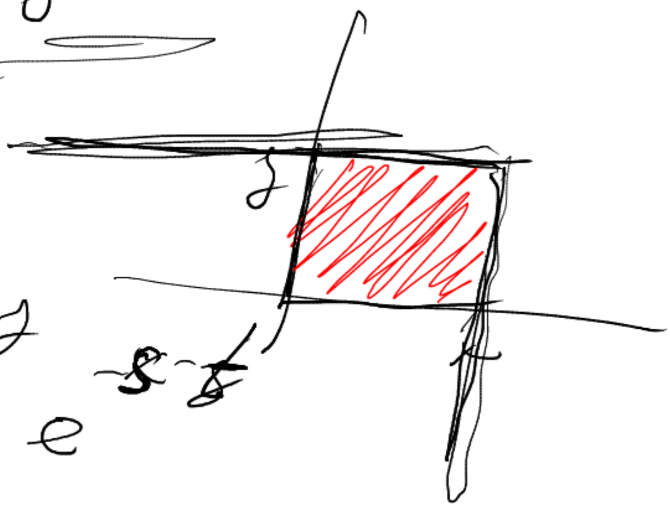
$$P((X,Y) \in B) = \iint_{(x,y) \in B} f_{X,Y}(x,y)$$

$$= \int_{x \in A} \overbrace{\int_{y \in \mathbb{R}} f_{X,Y}(x,y)}^{f_X(x)}$$

d) $F_X(x) = P(X \le x) = 1 - e^{-x}$ (více)

$$= \int_0^x e^{-s} =$$

$$F_Y(y) = 1 - e^{-y}$$




$$F_{X,Y}(x,y) = \int_0^x \int_0^y e^{-s-t}$$

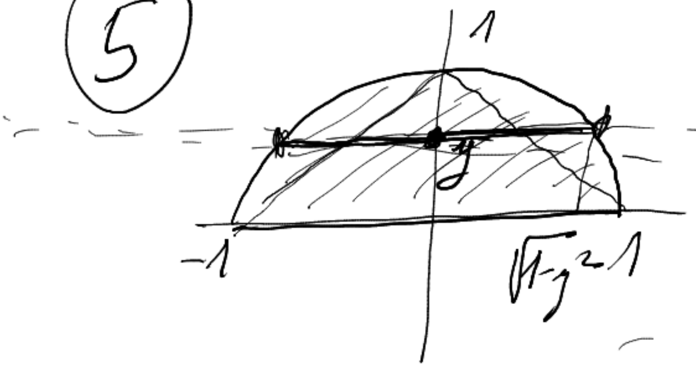
$$= \int_0^x e^{-s} \cdot \int_0^y e^{-t} =$$

$$= (1-e^{-x})(1-e^{-x})$$

$$= F_X(x)\, F_Y(y)$$

$\{X \le x\}$, $\{Y \le y\}$ jsou nez.

5



$\mathbb{E}X = 0$

$\frac{\pi \cdot 1^2}{2}$

a) $f_{X,Y}(x,y) = \begin{cases} \frac{2}{\pi} & x^2+y^2 \le 1,\ y > 0 \\ 0 & \text{jinak} \end{cases}$ $\quad \iint f = 1$

$\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$

b) $f_Y(y) = \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{2}{\pi} = \frac{4}{\pi}\sqrt{1-y^2}$

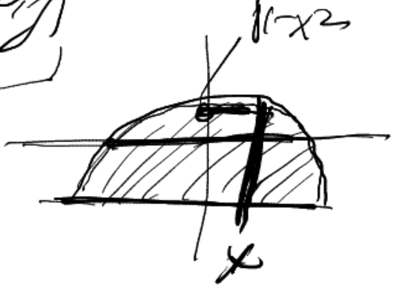
$\mathbb{E}(Y) = \int_0^1 y\, f_Y(y) = \frac{4}{\pi}\int_0^1 y\sqrt{1-y^2}$

$= \frac{-2}{\pi}\int_0^1 (-2y)\sqrt{1-y^2}$

$= \frac{-2}{\pi}\cdot\left[\frac{(1-y^2)^{3/2}}{3/2}\right]_0^1 = \frac{4}{3\pi}$

$\doteq 0.42$

c) $\mathbb{E}Y = \iint \underbrace{y}_{g(x,y)} \, f_{X,Y}(x,y) = \iint_{x,y} y \cdot \frac{2}{\pi}$

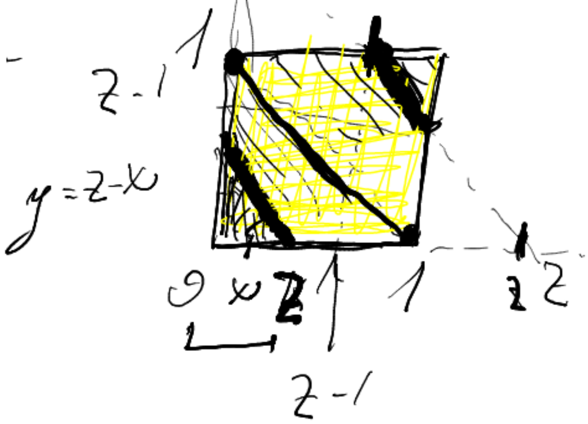


$$= \int_{-1}^{1} \underbrace{\int_{0}^{\sqrt{1-x^2}} y \cdot \frac{2}{\pi} \, dy}_{} \, dx$$

$$\left[\frac{1}{\pi} y^2\right]_0^{\sqrt{1-x^2}} = \frac{1-x^2}{\pi}$$

$$\int_{-1}^{1} \frac{1-x^2}{\pi} = \frac{2}{\pi} - \underbrace{\left[\frac{x^3}{3\pi}\right]_{-1}^{1}}_{\frac{+1+1}{3\pi}} = \frac{1}{\pi}\left(2 - \frac{2}{3}\right) = \frac{4}{3\pi}$$

(6) a) $X, Y \sim U(0,1)$ $\quad Z = X + Y$




$z \; \dots \; P(X+Y \le z)$

$= F_Z(z)$

$f_Z(z) = F_Z'(z)$

$F_Z(z) = \frac{1}{2} z^2 \rightsquigarrow f_Z(z) = \left(\frac{1}{2} z^2\right)' = z$

$z \in [0,1]$


$f_Z(z) = \int_x f_X(x) \, f_Y(z-x) \, dx$

$z \in [0,1]$: $\quad \int_0^z 1 \cdot 1 = z$ $\qquad x \in z-1 \dots$

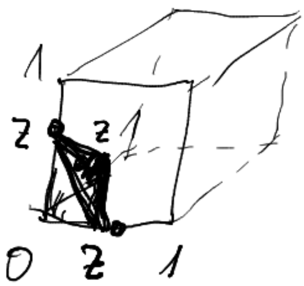
$z \in [1,2]$: $\quad \int_{z-1}^{1} 1 = 2 - z = 1-(z-1)$

$0 \le z - x \le 1$

$z-1 \le x \le z$

$0 \le x \le 1$

b)




$x + y + z = c \in [0,1]$

$F_S(s) = \frac{1}{6} z^3$

$f_S(s) = \frac{1}{2} z^2$