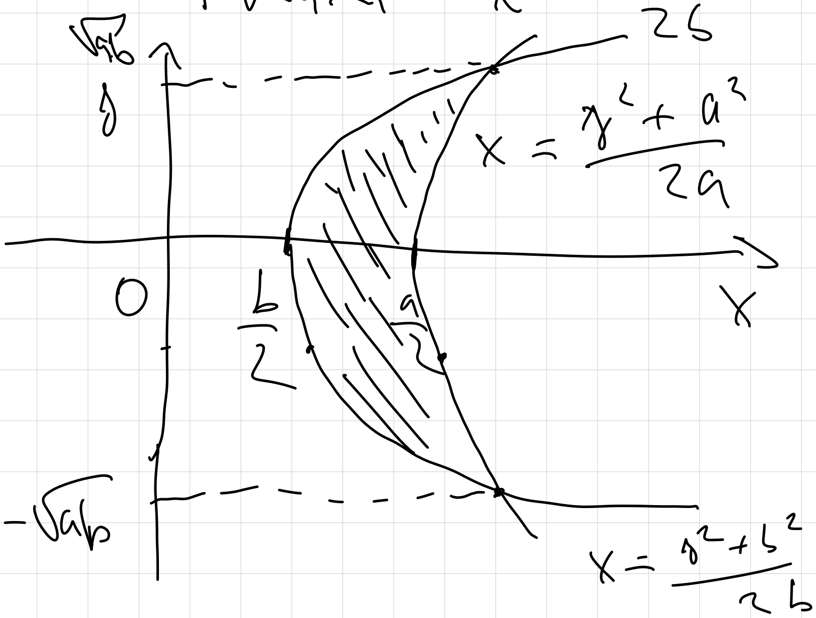
1) Určete míru $A \subseteq \mathbb{R}^2$ ohraničené křivkami $x = \dfrac{y^2+b^2}{2b}$ a $x = \dfrac{y^2+a^2}{2a}$ kde $a>b>0$.



$$A = \left\{ (x,y) \in \mathbb{R}^2 ;\ x \in \left(\frac{y^2+b^2}{2b}, \frac{y^2+a^2}{2a} \right) \right\}$$

- A je ot. $\Rightarrow$ A je měřitelná.

- $$\lambda^2(A) = \int_A 1 \, d\lambda^2(x,y) =$$

Fubini $(1 \geq 0)$

$$= \int_{-\sqrt{ab}}^{\sqrt{ab}} \int_{\frac{y^2+b^2}{2b}}^{\frac{y^2+a^2}{2a}} 1 \, dx \, dy =$$

$$= \int_{-\sqrt{ab}}^{\sqrt{ab}} \frac{y^2 + a^2}{2a} - \frac{y^2 + b^2}{2b} \, dy =$$

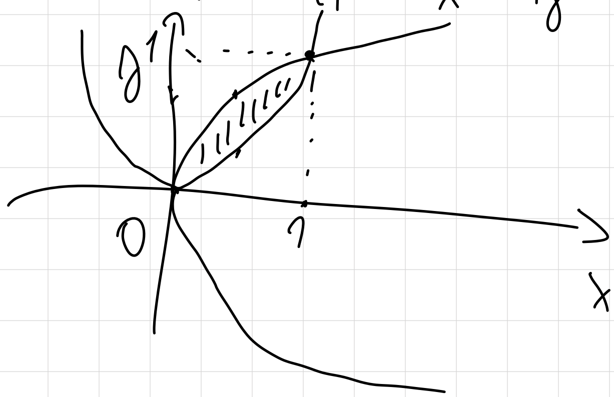
$$= \int_{-\sqrt{ab}}^{\sqrt{ab}} y^2 \frac{b - a}{2ab} + \frac{(a - b)}{b} \, dy =$$

$$\left[\frac{y^3}{3}\,\frac{b-a}{2ab}+y\,\frac{a-b}{2}\right]_{-\sqrt{ab}}^{\sqrt{ab}}=$$

$$=2\left(\frac{\sqrt{ab}\,(b-a)}{6}+\frac{\sqrt{ab}\,(a-b)}{2}\right)=$$

$$=\sqrt{ab}\,(a-b)\,\frac{2}{3}\,.$$

2) Spočítejte integrál z $f(x,y) = x^2 + y$ přes množinu $A \subseteq \mathbb{R}^2$ ohraničenou křivkami $x = y^2$, $y = x^2$:



$$A = \{(x,y) \in \mathbb{R}^2 ;\ x > y^2,\ y > x^2\}$$

- A je ot. $\Rightarrow$ A je měř.
- f je spoj. $\Rightarrow$ f je měř.

Fubini, $x^2+y \geq 0$ na A

$$\int_A x^2+y \, d\lambda^2(x,y) = \int_0^1 \int_{x^2}^{\sqrt{x}} x^2+y \, dy \, dx$$

$$= \int_0^1 \left[x^2 y + \frac{y^2}{2} \right]_{x^2}^{\sqrt{x}} dx =$$

$$\int_0^1 x^{\frac{5}{2}} - x^4 - \frac{1}{2}x^4 + \frac{1}{2}x \, dx =$$

$$= \left[\frac{2}{7}x^{\frac{7}{2}} - \frac{1}{5}x^5 - \frac{1}{10}x^5 + \frac{1}{4}x^2\right]_0^1 =$$

$$= \frac{2}{7} - \frac{3}{10} + \frac{1}{4} = \frac{40 - 42 + 35}{140} = \frac{33}{140}$$

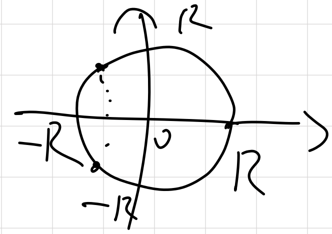
3) Spočtěte integrál $f(x,y) = y^2 \sqrt{R^2 - x^2}$

přes $A \subseteq \mathbb{R}^2$, $A = B_2(0, R) = \{(x,y) \in \mathbb{R}^2;\ x^2 + y^2 \leq R^2\}$

- A je uz. $\Rightarrow$ měř.
- f je spojitá na A, položím-li tedy

$\tilde{f} = f \cdot \chi_A$ resp. $\tilde{f} = f$ na A a

$\hat{f}$ je měř.



$$\int_A y^2 \sqrt{R^2 - x^2} \, d\lambda^2(x,y) =$$ Fubini, $f \geq 0$ na A.

$$= \int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} y^2 \sqrt{R^2 - x^2} \, dy \, dx =$$

$$= \int_{-R}^{R} \sqrt{R^2-x^2} \left[\frac{y^3}{3} \right]_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dx =$$

$$= \frac{2}{3} \int_{-R}^{R} (R^2-x^2)^2 \, dx = \frac{2}{3} \left[R^4 x - \frac{2}{3} R^2 x^3 \right.$$

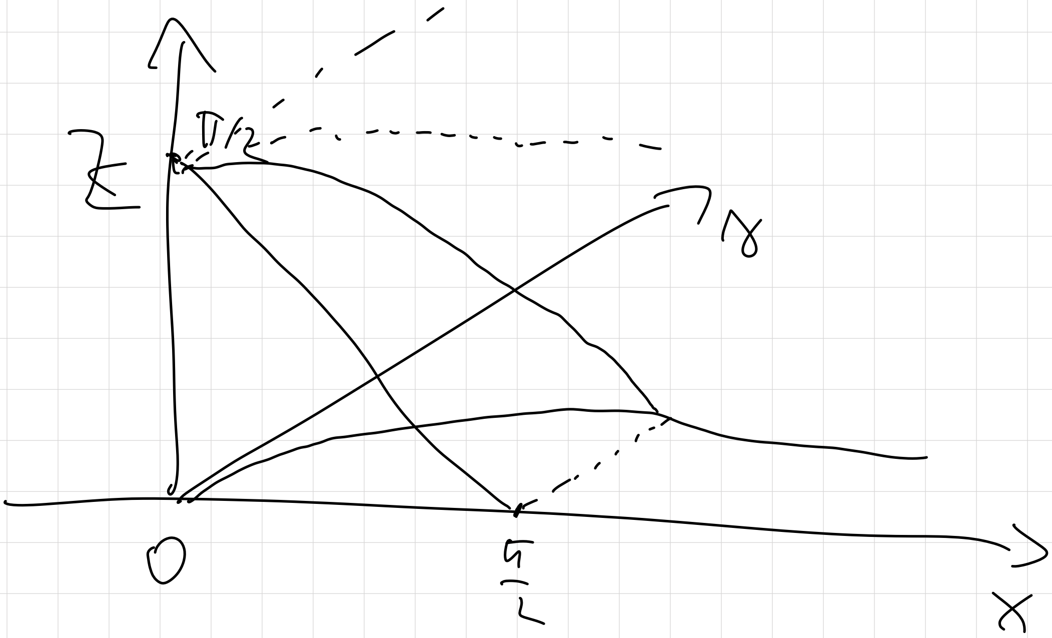
$$\left. + \frac{1}{5} x^5 \right]_{-R}^{R} = \frac{4}{3} R^5 \left(1 - \frac{2}{3} + \frac{1}{5} \right) =$$

$$= R^5 \frac{4}{3} \left(\frac{1}{3} + \frac{1}{5} \right) = \frac{32}{45} R^5$$

4) Spočítejte integrál $f(x,y,z) = y\cos(x+z)$

přes $A \subseteq \mathbb{R}^3$ ohraničenou plochami

$y = 0$, $z = 0$, $x+z = \frac{\pi}{2}$, $y = \sqrt{x}$, $x > 0$


z
π/2
y
0
π/2
x

$A = \{ (x, y, z) \in \mathbb{R}^3 ; \; y \in (0, \sqrt{x}), \; z \in (0, \frac{\pi}{2} - x),$
$x \in (0, \frac{\pi}{2}) \}$

A je ot. $\Rightarrow$ A je měř.

f je spoj. $\Rightarrow$ f je měř.

$$\int_A f(x,y,z)\, d\lambda^3(x,y,z) \overset{\text{Fubini } (*)}{=}$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{x}} \int_0^{\frac{\pi}{2}-x} y^2 \cos(x+z)\, dz\, dy\, dx$$

$(*)$ $\lambda^3(A) < \infty$, $|f|$ je na A omez.

$\Rightarrow \int_{\mathbb{R}^3} |f|\, \chi_A \, d\lambda^3 < \infty. \Rightarrow \int_A f \, d\lambda^3$ ex.

$$A \subseteq \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) = B$$

$$\lambda^3 \left(B \right) = \frac{\pi}{2} \cdot \frac{\pi}{2} \cdot \frac{\pi}{2} < \infty.$$

$$A \subseteq B \Rightarrow \lambda^3 (A) \leq \lambda^3 (B) < \infty.$$