mma 101 – 14. cvičení 18.11.20

1) $$\lim_{x\to 0} \frac{\sin 5x - \sin 3x}{\sin x} = \lim_{x\to 0} \frac{1}{\frac{\sin x}{x}} \left(\frac{\sin 5x}{5x}\cdot 5 - \frac{\sin 3x}{3x}\cdot 3 \right)$$

$$\overset{AL}{=} \frac{\lim\limits_{x\to 0} \frac{\sin 5x}{5x}\cdot 5 - \lim\limits_{x\to 0} \frac{\sin 3x}{3x}\cdot 3}{\lim\limits_{x\to 0} \frac{\sin x}{x}} = \frac{5-3}{1} = 2,$$

protože $\lim\limits_{x\to 0} \frac{\sin(kx)}{kx} = 1$, pro $k \in \mathbb{R}\setminus\{0\}$.

(Podle věty o limitě složené fce:

vnější: $f(y) = \frac{\sin y}{y} \xrightarrow[y\to A=0]{} 1$

vnitřní: $g(x) = kx \xrightarrow[x\to 0=:c]{} 0 =: A$

(P) $g(x) = 0 \iff x = 0$)

2) $$\lim_{x\to 0} \frac{\cos(xe^{x}) - \cos(xe^{-x})}{x^3}$$

Jak na? $$\lim_{x\to 0} \frac{\cos(xe^{x}) - 1 + 1 - \cos(xe^{-x})}{x^2} \cdot \frac{1}{x} =$$

$$= \lim_{x\to 0} \underbrace{\left[\underbrace{\frac{\cos(xe^{x}) - 1}{x^2 e^{2x}}}_{\to -\frac{1}{2}} \underbrace{e^{2x}}_{\to 1} + \underbrace{\frac{1-\cos(xe^{-x})}{x^2 e^{-2x}}}_{\to \frac{1}{2}} \cdot \underbrace{e^{-2x}}_{\to 1} \right]}_{\to 0 \ x\to 0} \frac{1}{X}$$

nelze použít AL: $[\quad] \xrightarrow[x\to 0]{} 0$ a $x \to 0$, $x \to 0$.

$$[\,]: \lim_{x\to 0} \frac{\cos(xe^x)-1}{(xe^x)^2} = -\lim_{x\to 0} \frac{1-\cos(xe^x)}{(xe^x)^2} \underset{\text{VoLS}}{=} -\frac{1}{2}$$

- vnější: $f(y) = \dfrac{1-\cos y}{y} \xrightarrow[y\to A=0]{} \dfrac{1}{2}$
- vnitřní: $g(x) = xe^x \xrightarrow[x\to 0=:C]{} 0=:A$
- (P) $xe^x = 0 \iff x = 0$; $y = 0$ na lib. $P(0)$

$$\lim_{x\to 0} \frac{\cos(xe^x)-\cos(xe^{-x})}{x^3} = (++)$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\cos(\alpha-\beta) - \cos(\alpha+\beta) = 2\sin\alpha\sin\beta$$

$$\left[\begin{aligned} \alpha-\beta &= xe^x \\ \alpha+\beta &= xe^{-x} \end{aligned}\right. \qquad \begin{aligned} 2\alpha &= xe^x + xe^{-x} \\ 2\beta &= xe^{-x} - xe^x \end{aligned}$$

$$= \lim_{x\to 0} 2 \cdot \frac{\sin\left(x\frac{e^x+e^{-x}}{2}\right) \cdot \sin\left(x \cdot \frac{e^{-x}-e^x}{2}\right)}{x^3} =$$

$$= 2\lim_{x\to 0} \underbrace{\frac{\sin\left(x\frac{e^x+e^{-x}}{2}\right)}{x\frac{e^x+e^{-x}}{2}}}_{(\mathrm{I})\ \to 1;\, x\to 0} \cdot \frac{e^x+e^{-x}}{2} \cdot \underbrace{\frac{\sin\left(x\frac{e^{-x}-e^x}{2}\right)}{x\frac{e^{-x}-e^x}{2}}}_{\mathrm{II}} \cdot \underbrace{\frac{e^{-x}-e^x}{2x}}_{\mathrm{III}}$$

k(I): VoLS: vnější: $f(y) = \frac{\sin y}{y} \xrightarrow[y \to A = 0]{} 1$

vnitřní: $g(x) = x\,\frac{e^x + e^{-x}}{2} \xrightarrow[x \to 0 =: c]{} 0 =: A$

(P) $g(x) = 0 \Leftrightarrow x = 0$

k(II): $g(x) = x\,\frac{e^{-x} - e^x}{2}$

$g(x) = 0$; $e^{-x} = e^x$; $e^{-x} - e^x = 0$;

$e^{-x}(1 - e^{2x}) = 0 \Leftrightarrow x = 0$

$$\lim_{x \to 0} \frac{e^{-x} - e^x}{2x} = \lim_{x \to 0} \underbrace{\frac{e^x(e^{-2x} - 1)}{-2x}}_{\xrightarrow[x \to 0]{} 1} (-1) = -1$$

vnější: $f(y) = \frac{e^y - 1}{y} \xrightarrow[y \to A = 0]{} 1$

vnitřní: $g(x) = -2x \xrightarrow[x \to c := 0]{} 0 =: A$

(P) $g(x) = 0 \Leftrightarrow x = 0$

Závěr: $(+\,+) \overset{AL}{=} -2$

K(8) : $\sin(nx) = \sin(\underbrace{n(x-\pi)}_{\alpha} + \underbrace{\pi n}_{\beta}) =$

$$\sin(n(x-\pi))\underbrace{\cos(n\pi)}_{(-1)^n} + \cancel{\cos(n(x-\pi))}\underbrace{\sin(n\pi)}_{=0}$$

K(5): $\sqrt{x+\sqrt{x+\sqrt{x}}} = \sqrt{x}\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{\sqrt{x^2}}} =$

$$= \sqrt{x}\sqrt{1+\sqrt{\underbrace{\frac{1}{x}+\frac{1}{x^{3/2}}}_{\xrightarrow[x\to+\infty]{} 0}}}$$