4. Funkce jedné reálné proměnné – derivace a Taylorův polynom

4.1. Derivace funkce

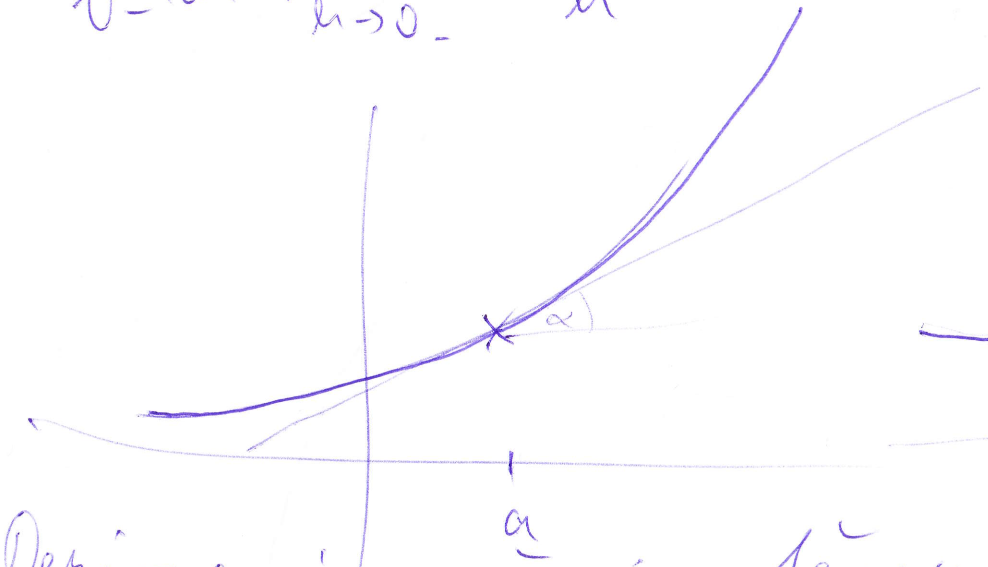
Def Nechť f je reálná funkce a $a \in \mathbb{R}$. Pak derivací f v bodě a budeme rozumět $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$,

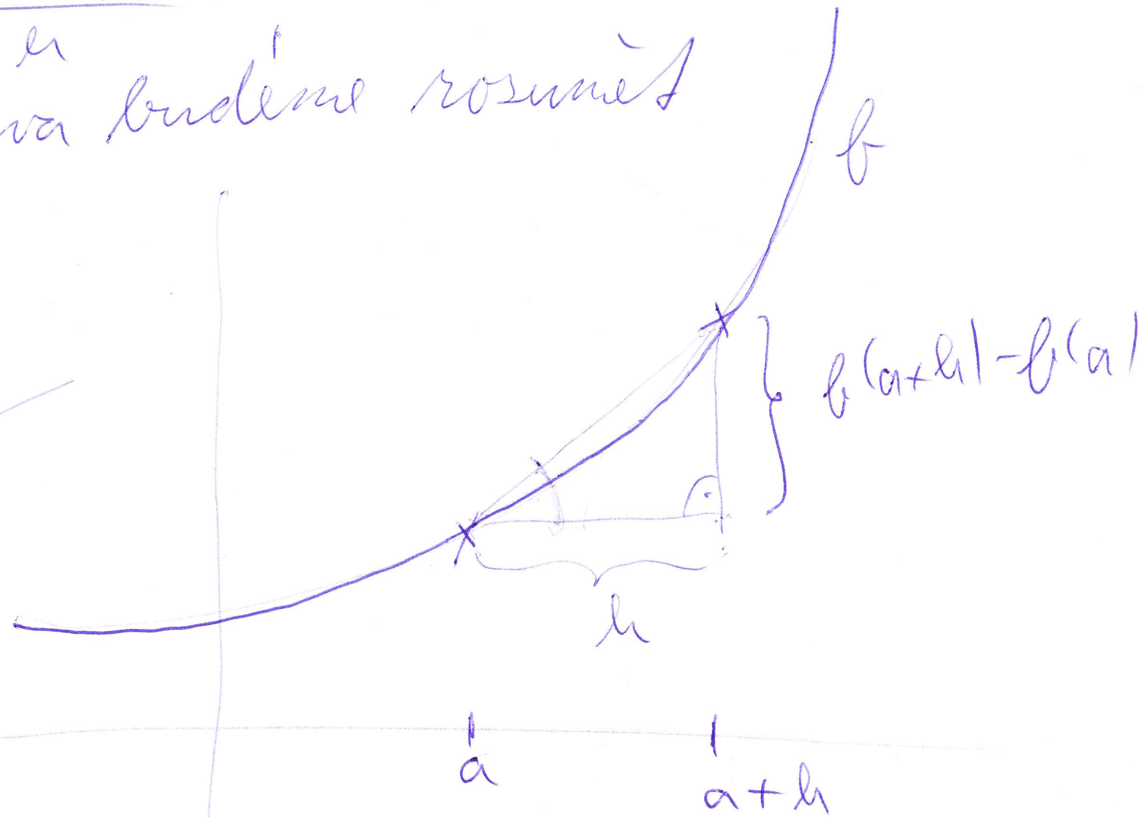
derivací f v bodě a zprava budeme rozumět

$$f'_+(a) = \lim_{h \to 0+} \frac{f(a+h) - f(a)}{h}$$

derivací f v bodě a zleva budeme rozumět

$$f'_-(a) = \lim_{h \to 0-} \frac{f(a+h) - f(a)}{h}$$





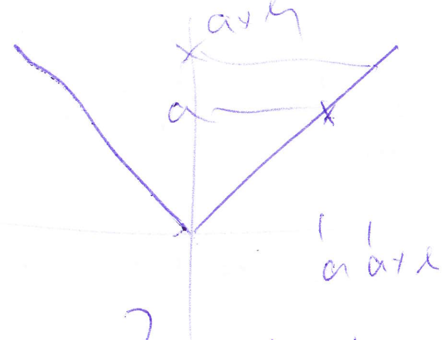
Derivace je směrnice tečny.

Poznámky: 1) $f'(a)$ $\begin{cases} \text{existuje} \begin{cases} \text{vlastní} & f'(a) \in \mathbb{R} \\ \text{nevlastní} & f'(a) \in \{+\infty, -\infty\} \end{cases} \\ \text{neexistuje} \end{cases}$

2) $$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

3) $$f'(a) = A \iff \left(f'_+(a) = A \ \& \ f'_-(a) = A \right)$$

Příklady: 1) $f(x) = |x|$

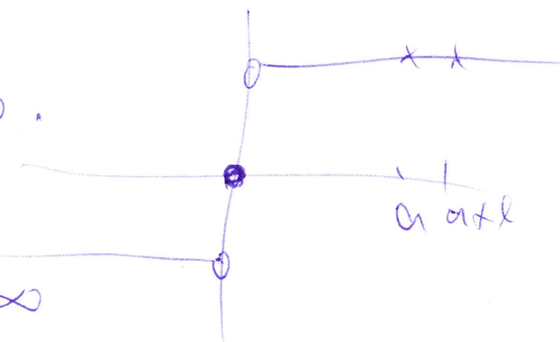


$a > 0$ $$f'(a) = \lim_{h \to 0} \frac{a+h-a}{h} = \lim_{h \to 0} 1 = 1$$

analogicky $f'(a) = -1$ pro $a < 0$

$$\left. \begin{aligned} f'_+(0) &= \lim_{h \to 0+} \frac{|0+h| - |0|}{h} = \lim_{h \to 0+} \frac{h}{h} = 1 \\ f'_-(0) &= \lim_{h \to 0-} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{-h}{h} = -1 \end{aligned} \right\} f'(0) \text{ neex.}$$

2) $f(x) = \operatorname{sgn} x$



$$f'_+(0) = \lim_{h \to 0+} \frac{\operatorname{sgn}(h) - \operatorname{sgn}(0)}{h} = \lim_{h \to 0+} \frac{1 - 0}{h} = +\infty.$$

$$f'_-(0) = \lim_{h \to 0-} \frac{\operatorname{sgn} h - \operatorname{sgn}(0)}{h} = \lim_{h \to 0-} \frac{-1 - 0}{h} = +\infty$$

$\exists\, f'(0) = +\infty$

$f'(a) = 0 \quad \forall a \neq 0$

3) $f(x) = x^n$ $\qquad f'(x) = n \cdot x^{n-1}$

$\underline{n=2}$ $\quad f'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = \lim\limits_{h\to 0} \dfrac{(a+h)^2-a^2}{h} =$

$$= \lim_{h\to 0} \frac{\cancel{a^2}+2ah+h^2-\cancel{a^2}}{h} = \lim_{h\to 0} 2a + \lim_{h\to 0} h = 2a+0$$

$\underline{n\in\mathbb{N}}$: $\quad f'(a) = \lim\limits_{h\to 0} \dfrac{(a+h)^n-a^n}{h} =$

$$= \lim_{h\to 0} \frac{\cancel{a^n}+\binom{n}{1}\cdot a^{n-1}\cdot h+\binom{n}{2}\cdot a^{n-2}\cdot h^2+\dots+h^n-\cancel{a^n}}{h}$$

$$= \lim_{h\to 0} n\cdot a^{n-1} + \lim_{h\to 0}\binom{n}{2}\cdot a^{n-2} h + \quad \dots \quad + \lim_{h\to 0} h^{n-1} =$$

n× aritmetiky limit (uzavřené) $\qquad = n a^{n-1} + 0 + \dots + 0 = n\cdot a^{n-1}$

$\boxed{\text{Věta L 4.1}}$ (vztah derivace a spojitosti) Nechť má funkce f v bodě $a\in\mathbb{R}$ derivaci $f'(a)\in\mathbb{R}$. Pak je f v bodě a spojitá.

$\underline{\text{Dk:}}$ $\quad \lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} \left(\dfrac{f(x)-f(a)}{x-a}\right)(x-a) + f(a) =$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \cdot \lim_{x\to a}(x-a) + \lim_{x\to a} f(a) = \underbrace{f'(a)}_{\in\mathbb{R}}\cdot 0 + f(a) = f(a)$$

□

$\boxed{\text{Věta T 4.2}}$ (aritmetika derivací). Nechť $f'(a)$ a $g'(a)$ existují.

(i) $(f+g)'(a) = f'(a) + g'(a)$, pokud má pravá strana smysl.

(ii) Nechť je g spojitá v a, pak $(f\cdot g)'(a) = f'(a)\cdot g(a) + f(a)\cdot g'(a)$, pokud má pravá strana smysl.

(iii) Nechť je g spojitá v a a $g(a) \neq 0$, pak
$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\cdot g(a) - f(a)\cdot g'(a)}{g^2(a)}$, pokud má pravá strana smysl.

Dk: (i)

$$(f+g)'(a) = \lim_{h\to 0} \frac{f(a+h) + g(a+h) - (f(a) + g(a))}{h} =$$

$$\overset{?}{=} \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} = f'(a) + g'(a).$$

$\boxed{17-5}$

(ii) $(f\cdot g)'(a) = \lim\limits_{h\to 0} \dfrac{f(a+h)\cdot g(a+h) - f(a)\cdot g(a) - f(a)\cdot g(a+h) + f(a)\cdot g(a+h)}{h}$

$\overset{?}{=} \lim\limits_{h\to 0} \dfrac{f(a+h)\cdot g(a+h) - f(a)\cdot g(a+h)}{h} + \lim\limits_{h\to 0} \dfrac{f(a)\cdot g(a+h) - f(a)\cdot g(a)}{h}$

$\overset{?}{=} \lim\limits_{h\to 0} g(a+h)\cdot \lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} + \lim\limits_{h\to 0} f(a)\cdot \lim\limits_{h\to 0} \dfrac{g(a+h)-g(a)}{h}$

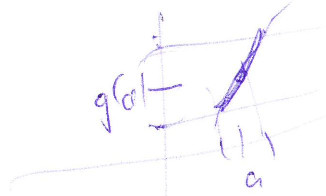
g je spojitá v a

$= g(a) \cdot f'(a) + f(a)\cdot g'(a)$.

(iii) Víme, že $g(a)\neq 0$ a g je spojitá v a.

Tedy ~~$\exists\delta>0$~~ $\exists \delta > 0$ tak, že $g(a+h)\neq 0$ $\forall h\in B(0,\delta)$

(~~A~~ $g(a)>0$ a $g(a)\in\mathbb{R}$. K $\varepsilon = \frac{g(a)}{2}$ $\exists\delta>0$ $\forall h\in B(0,\delta)$: $|g(a+h)-g(a)| < \varepsilon = \frac{g(a)}{2}$)



(iii) $\exists \delta > 0 \;\; g(a+h) \neq 0 \;\; \forall h \in B(0,\delta)$.

$$\left(\frac{f}{g}\right)'(a) = \lim_{h \to 0} \frac{\frac{f(a+h)}{g(a+h)} - \frac{f(a)}{g(a)}}{h} = \lim_{h \to 0} \frac{f(a+h)\cdot g(a) - f(a)\cdot g(a+h)}{g(a+h)\cdot g(a)\cdot h}$$

$$\stackrel{*}{=} \lim_{h \to 0} \frac{1}{g(a+h)\cdot g(a)} \cdot \lim_{h \to 0} \frac{f(a+h)\cdot g(a) - f(a)\cdot g(a+h) + f(a)g(a) - f(a)g(a)}{h}$$

$$\stackrel{*}{=} \frac{1}{g^2(a)} \cdot \left(\lim \frac{f(a+h)\cdot g(a) - f(a)\cdot g(a)}{h} + \lim_{h \to 0} \frac{f(a)\cdot g(a) - f(a)\cdot g(a+h)}{h}\right)$$

g spojitá v a

$$= \frac{1}{g^2(a)} \cdot \left(\lim_{h \to 0} g(a) \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} f(a) \cdot \lim_{h \to 0} \frac{g(a) - g(a+h)}{h}\right)$$

$$= \frac{1}{g^2(a)} \cdot \left(g(a) \cdot f'(a) + f(a) \cdot (-g'(a))\right) \qquad \square$$

$\boxed{\text{Věta T 4.3}}$ (derivace složené funkce) $\boxed{17-7}$

Nechť má f derivaci v bodě y_0, g má derivaci v x_0, je v x_0 spojitá a $y_0 = g(x_0)$. Pak

$$(f \circ g)'(x_0) = f'(y_0) \cdot g'(x_0) = f'(g(x_0)) \cdot g'(x_0),$$

je-li výraz vpravo definován.

$$\boxed{(f \circ g)(x) = f(g(x))}$$

~~Dk.~~ Myšlenka dk:

$$\lim_{h \to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{h} =$$

$$= \lim_{h \to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{g(x_0+h) - g(x_0)} \cdot \frac{g(x_0+h) - g(x_0)}{h}$$

VOLSF $\downarrow \frac{f(y) - f(y_0)}{y - y_0}$ $\qquad \downarrow g'(x_0)$

$f'(g(x_0))$

PROBLÉMY: co když $g(x_0+h) = g(x_0)$?

Příklady: 1) $\left(e^{x^2+x}\right)' = \left(f(g(x))\right)'$ $\qquad g(x) = x^2+x \qquad f(z) = e^z$ $\qquad \boxed{17-8}$

$f'(z) = e^z$

$= f'(g(x)) \cdot g'(x) = e^{x^2+x} \cdot (x^2+x)' =$

$= e^{x^2+x} \cdot \left((x^2)' + x'\right) = e^{x^2+x} \cdot (2x+1).$

2) $(x^n)' = n \cdot x^{n-1} \qquad (x^a)' \qquad a > 0$

$x^a = e^{a \cdot \log x}$

$(x^a)' = \left(e^{a \cdot \log x}\right)' = \left(f(g(x))\right)' =$

$g(x) = a \cdot \log x, \quad f(z) = e^z, \quad f'(z) = e^z$

PŘÍŠTĚ BUDE TABULKA DERIVACÍ!

$(e^x)' = e^x$ „dlouhá“

$(\log x)' = \frac{1}{x}$

$= f'(g(x)) \cdot g'(x) = e^{a \cdot \log x} \cdot (a \cdot \log x)' =$

$= x^a \cdot (\underbrace{a'}_{=0} \cdot \log x + a \cdot (\log x)') = x^a \cdot a \cdot \frac{1}{x} = a \cdot x^{a-1}$

$(\sqrt{x})' = \frac{1}{2} \cdot x^{\frac{1}{2}-1} = \frac{1}{2} \frac{1}{\sqrt{x}}$