Sečtěte řadu

$$\sum_{n=0}^{\infty} \frac{n(n+1)}{2^n}$$

Uvažme mocninou řadu $f(x) = \sum_{n=0}^{\infty} x^n$

$$f'(x) = \sum_{n=0}^{\infty} n\, x^{n-1}$$

$$f''(x) = \sum_{n=0}^{\infty} n(n-1)\, x^{n-2}$$

Zřejmě poloměry konvergencí všech tří řad jsou rovny 1, tedy věta byla použita správně.

Protože $f(x) = \frac{1}{1-x}$ dostáváme

$$f'(x) = \frac{1}{(1-x)^2}, \quad f''(x) = 2\frac{1}{(1-x)^3}$$

Protože $\sum_{n=0}^{\infty} n(n+1)\, x^n$ má poloměr konvergence 1, můžeme ji vyjádřit jako

$$\sum_{n=0}^{\infty} n(n+1)\, x^n = x^2 \sum_{n=0}^{\infty} n(n-1)\, x^{n-2} + 2x \sum_{n=0}^{\infty} n\, x^{n-1} = \frac{2x^2}{(1-x)^3} + \frac{2x}{(1-x)^2}$$

pro $\forall x \in (-1, 1)$

položme $x = \frac{1}{2} \in (-1,1)$. Pak

$$\sum_{n=0}^{\infty} \frac{n(n+1)}{2^n} = \frac{2\cdot\left(\frac{1}{2}\right)^2}{\left(1-\frac{1}{2}\right)^3} + \frac{2\cdot\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}$$

$$= 4 + 4 = 8$$

$$n(n+1)x^n = n^2 x^n + n x^n = c_n$$

$$\underline{x^2}\left(n(n-1)x^{n-2}\right) = \left(n^2 x^{n-2} - n x^{n-2}\right)x^2$$

$$= n^2 x^n - n x^n = a_n$$

$$\underline{2x}\left(n x^{n-1}\right) = 2n x^n = b_n$$

$$a_n + b_n = n^2 x^n + n x^n = c_n$$

Sečtěte
$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} \cdot 1$$

Uvažme
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{4n^2-1} x^{2n+1}$$

Pak pol. konv. je 1 a $f(1)$ konverguje.

Poloměr konvergence:

$$\sum_{n=1}^{\infty} \frac{1}{4n^2-1} x^{2n+1} = \sum_{n=0}^{\infty} b_n x^n$$

$$\limsup_{n\to\infty} \sqrt[n]{b_n} = \limsup_{n\to\infty} \sqrt[n]{\frac{1}{4n^2-1}} = 1$$

$\Rightarrow$ Podle Abelovy věty platí

$$F(1) = \lim_{x \to 1_-} f(x).$$

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n} = x \underbrace{\sum_{n=1}^{\infty} \frac{1}{2n-1} x^{2n-1}}_{g(x)}$$

pro $\forall x \in (-1, 1)$

Polomer konv. g je 1.

$$g'(x) = \sum_{n=1}^{\infty} x^{2n-2} \overset{(*)}{=} x^{-2} \sum_{n=1}^{\infty} (x^2)^n =$$

$$= x^{-2}\, x^2 \frac{1}{1-x^2} = \frac{1}{1-x^2}$$

$(*)$ platí na $(-1,1)\setminus\{0\}$, v 0 triv.

$$\sum_{n=1}^{\infty} 0^{2n-2} = 1 = \frac{1}{1-0^2}.$$

Integrací $g(x) = \int \frac{1}{1-x^2}\,dx =$

$$= \int \frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x-1}\,dx =$$

$$= \frac{1}{2}\log(x+1) - \frac{1}{2}\log(1-x) + C$$

$g(0) = 0 \Rightarrow C = 0.$

$$\Rightarrow F'(x) = x\,g(x) = \frac{1}{2}\left(x\log(x+1) - x\log(1-x)\right)$$

$$\Rightarrow F(x) = \frac{1}{2}\left(\int x\log(x+1)\,dx - \int x\log(1-x)\,dx\right)$$

$$= \frac{1}{2}\left(\overbrace{\frac{x^2}{2}\log(x+1)}^{A(x)} - \int\frac{x^2}{2x+2}\,dx - \overbrace{\frac{x^2}{2}\log(1-x)}^{B(x)}\right.$$

$$\left. - \int\frac{x^2}{2(1-x)}\,dx\right) =$$

$$= \frac{1}{2}\Bigg(A(x) - B(x) - \frac{1}{2}\int \underbrace{\frac{x^2-1}{x+1}}_{(x-1)} + \frac{1}{x+1}\,dx$$

$$- \frac{1}{2}\int \overbrace{\frac{x^2-1}{1-x}}^{-(x+1)} + \frac{1}{1-x}\,dx \Bigg) =$$

$$= \frac{1}{2}\Bigg(\frac{x^2}{2}\log(x+1) - \frac{x^2}{2}\log(1-x) - \frac{1}{2}\Big[\frac{x^2}{2} - x + \log(x+1)\Big] - \frac{1}{2}\Big\{-\frac{x^2}{2} - x - \log(1-x)\Big\}\Bigg)$$

$+\tilde{C}$

Je liko ž $f(x) = 0$, dostáváme

$$0 = \tilde{C} + 0$$

$$f(x) = \frac{1}{2}\left[\frac{x^2 \log(x+1)}{2} - \frac{x^2}{4} + \frac{x}{2} - \frac{1}{2}\log(x+1) + \frac{x^2}{4} + \frac{x}{2} + \log(x-1)\left(-\frac{1}{2}\right)(x-1)\cdot(x+1)\right]$$

Protože $\log(y) \cdot y \xrightarrow{y \to 0} 0$ dostáváme

$$f(1) = \lim_{x \to 1_-} f(x) = \frac{1}{2}\left(\frac{\log(2)}{2} - \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{2} - \frac{\log(2)}{2}\right)$$

$$= \frac{1}{2}.$$