V 3.12 Existuje funkce $\exp: \mathbb{R} \to \mathbb{R}$:

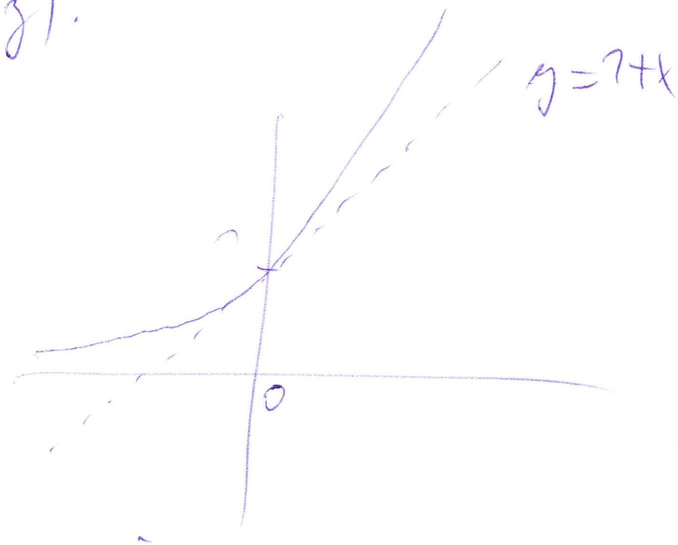
a) $\exp$ je rostoucí na $\mathbb{R}$

b) $\forall x, y \quad \exp(x+y) = \exp(x) \cdot \exp(y)$.

c) $\exp(0) = 1$

d) $\lim\limits_{x \to 0} \dfrac{\exp(x) - 1}{x} = 1$

e) $\exp x$ je spojitá na $\mathbb{R}$



Idea důk.: $e^x = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!} = 1 + x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} + \dots$

$\exp: \mathbb{R} \to (0, \infty)$

V 3.11 (o inverzní funkci)

Nechť f je spojitá a rostoucí na intervalu J.

Potom je funkce f^{-1} spojitá a rostoucí na intervalu $f(J)$.

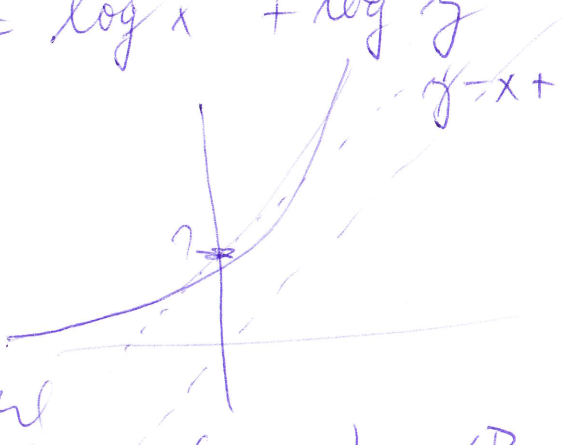
Def Funkci inverzní k exponenciele exp je logaritmus log.

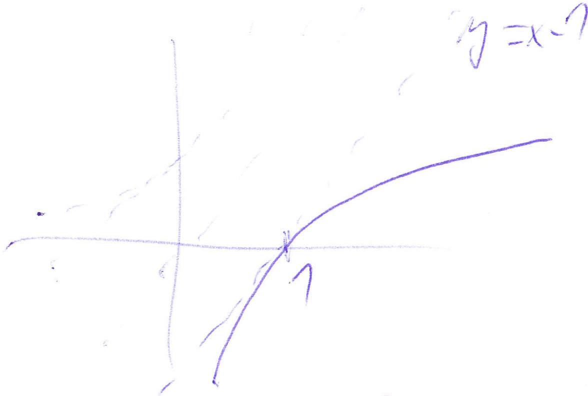
$\boxed{\text{Věta T 3.13}}$ (vlastnosti logaritmu) Funkce log splňuje:

a) $\log : (0,\infty) \to \mathbb{R}$ je spojitá a rostoucí funkce.

b) $\forall x,y>0: \quad \log(x\cdot y) = \log x + \log y$

c) $\lim\limits_{x \to 1} \dfrac{\log x}{x-1} = 1$





Dk: a)

exp je spojitá a rostoucí

$\Rightarrow$ existuje inverzní funkce

$\exp: \mathbb{R} \to (0,\infty) \Rightarrow \log: (0,\infty) \to \mathbb{R}$

Podle V 3.11 o inverzní funkci je log spojitá a rostoucí funkce.

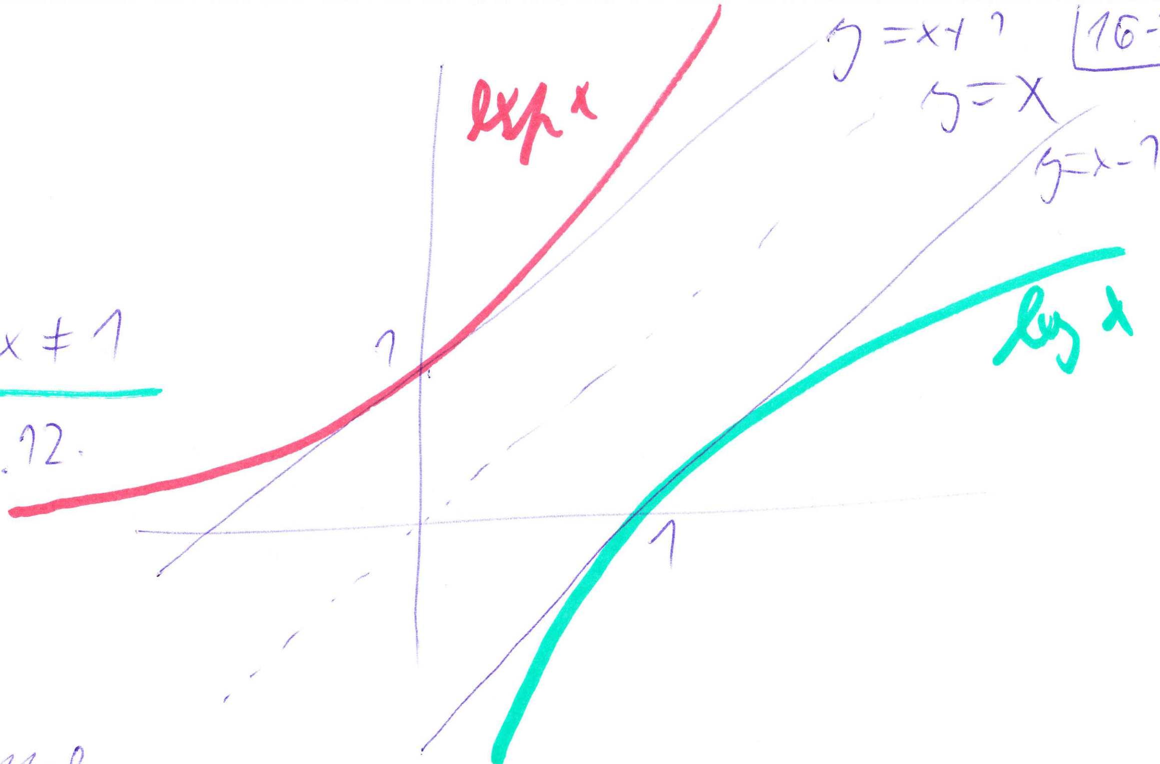
b) $\log x = A \iff \exp(A) = x$, $\quad \log y = B \iff \exp(B) = y$

z V 3.12 b):

$x\cdot y = \exp(A)\cdot\exp(B) \overset{b)}{=} \exp(A+B)$

$\Rightarrow \log(x\cdot y) = A + B = \log x + \log y$

c) $\lim\limits_{x\to 1} \frac{\log x}{x-1} = 1$



$f(y) = \frac{\exp y - 1}{y}$ $\quad y \neq 0$

$g(x) = \log x$, pak $g(x) \neq 0 \ \forall x \neq 1$

víme $\lim\limits_{y\to 0} f(y) = 1$ z V 3.12.

a $\lim\limits_{x\to 1} g(x) = 0$

Podle VOLSF (P) dostaneme

$$1 = \lim_{x\to 1} f(g(x)) = \lim_{x\to 1} \frac{\exp(\log x) - 1}{\log x} = \lim_{x\to 1} \frac{x-1}{\log x} \qquad \square$$

$$\lim_{x\to 1} \frac{\log x}{x-1} = 1 \iff \lim_{x\to 0} \frac{\log(1+x)}{x} = 1$$

Def: Nechť $a > 0$ a $b \in \mathbb{R}$. Pak definujeme $a^b = \exp(b \cdot \log(a))$.
Je-li $b > 0$, pak definujeme $\log_b a = \frac{\log a}{\log b}$.

Pozn.: $\overset{x>0}{x^2} = x^2 = \exp(2\cdot\log x) = \exp(\log x + \log x) = \exp(\log x) \cdot \exp(\log x) = x \cdot x$

$x^2 = x \cdot x$

Příklad: $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \overset{\text{Heine}}{=} \lim_{x\to 0}(1+x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\cdot\log(1+x)}$ (16-4

LIMITA TYPU 1^∞ $\quad f(z)=e^z \quad g(x)=\frac{1}{x}\log(1+x) \quad$ "$\lim_{x\to 0} f(g(x))$"

$$\underset{e^z \text{ spojitá}}{\overset{\text{VOLSF}}{=}} e^{\lim_{x\to 0}\frac{\log(1+x)}{x}} = e^1 = e.$$

$p\in\mathbb{R}$, $p\neq 0$

$$\lim_{n\to\infty}\left(1+\frac{p}{n}\right)^n \overset{\text{Heine}}{=} \lim_{x\to 0}(1+p\cdot x)^{\frac{1}{x}} = \lim_{x\to 0} e^{\frac{1}{x}\cdot\log(1+px)} \underset{\exp z \text{ je spojitá}}{\overset{\text{VOLSF}}{=}} e^{\lim_{x\to 0}\frac{\log(1+px)}{x}} = e^p$$

$$\lim_{x\to 0}\frac{\log(1+px)}{p\cdot x}\cdot p \underset{\tilde{g} \text{ splňuje (P)}}{\overset{\text{VOLSF}}{=}} 1\cdot p$$

$\tilde{f}(z)=\frac{\log(1+z)}{z} \xrightarrow{z\to 0} 1$, $\quad \tilde{g}(x)=p\cdot x \quad \tilde{g}(x)\neq 0 \ \forall x\neq 0$

POSTUP na 1^∞:

(0. Heine)

1. $a^b = e^{b\cdot\log a}$
2. $\lim e^{b\cdot\log a} \underset{\text{VOLSF (S)}}{=} e^{\lim b\cdot\log a}$
3. zbavíme se log pomocí $\lim\frac{\log(1+x)}{x}$
4. dopočteme.

$\boxed{\text{Věta BD 3.14}}$ (zavedení sinu a cosinu)

Existují funkce $\sin: \mathbb{R} \to \mathbb{R}$ a $\cos: \mathbb{R} \to \mathbb{R}$ splňující

a) $\forall x, y \in \mathbb{R}$ $\sin(x+y) = \sin x \cdot \cos y + \cos x \cdot \sin y$

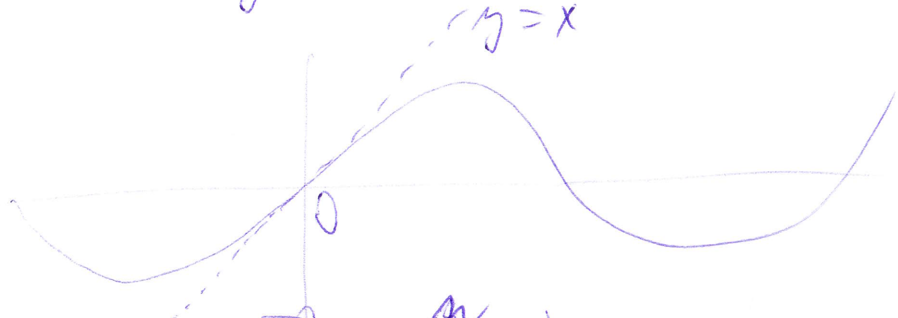
$\cos(x+y) = \cos x \cdot \cos y - \sin x \cdot \sin y$

$\cos(-x) = \cos x$, $\sin(-x) = -\sin x$

b) existuje kladné číslo π tak, že $\sin$ je rostoucí na $[0, \frac{\pi}{2}]$

a $\sin(\frac{\pi}{2}) = 1$

c) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$



Idea dk.: $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!} \cdot (-1)^n$.

$\boxed{\sum_{n=0}^{\infty} a_n \cdot x^n}$

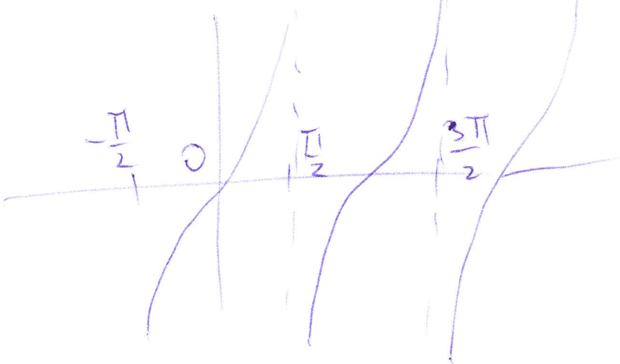
$\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \dots = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} \cdot (-1)^n$.

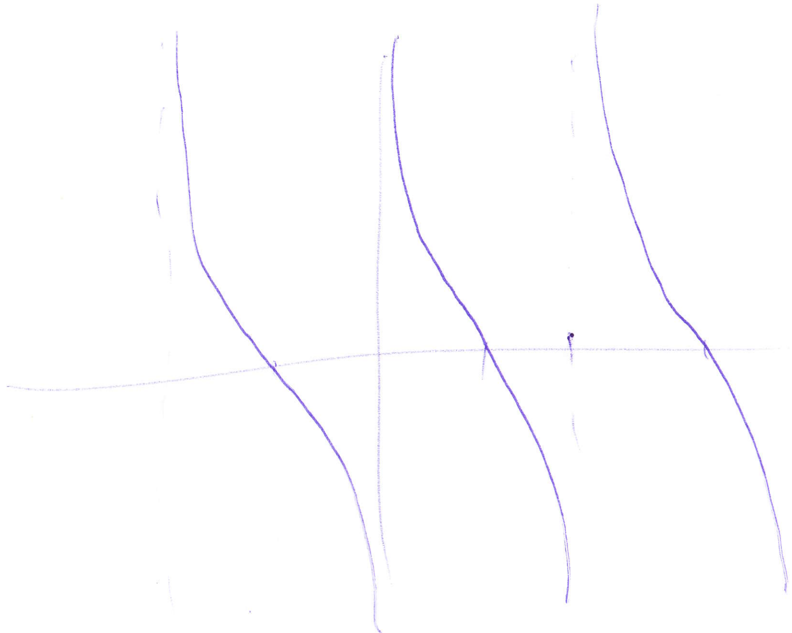
Příklad: $\lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} \left(= \frac{1}{2}\right) = \lim\limits_{x \to 0} \frac{1 - \cos x}{x^2} \cdot \frac{(1 + \cos x)}{1 + \cos x} = \lim\limits_{x \to 0} \frac{1 - \cos^2 x}{x^2 \cdot (1 + \cos x)}$

$\stackrel{?}{=} \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \lim\limits_{x \to 0} \frac{\sin x}{x} \cdot \lim\limits_{x \to 0} \frac{1}{1 + \cos x} = 1 \cdot 1 \cdot \frac{1}{1+1} = \frac{1}{2}$

Def Pro $x \in \mathbb{R} \setminus \{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\}$ a $y \in \mathbb{R} \setminus \{k \cdot \pi, k \in \mathbb{Z}\}$ definujeme funkce tangens a cotangens předpisem

$$\tan x = \frac{\sin x}{\cos x} \qquad \operatorname{cotg} y = \frac{\cos y}{\sin y} \qquad (\cancel{\text{cotan}})$$





Věta L 3.15 (spojitost sinu a cosinu).

Funkce sin, cos, tan a cotg jsou spojité na svém definičním oboru.

Dk: a) sin, pak b) $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ a složení ~~spojitého~~ spojitých funkcí je spojitá funkce

16 – 7

c) $\tan x = \frac{\sin x}{\cos x}$ $\cot g\, x = \frac{\cos x}{\sin x}$ jsou spojité z aritmetiky limit.

a) $\lim_{x \to a} \sin x \overset{?}{=} \sin a$

v 0: $\lim_{x \to 0} \sin x = \lim_{x \to 0} \frac{\sin x}{x} \cdot x \overset{\cancel{8}}{=} 1 \cdot 0 = \sin(0)$ ✓

$\lim_{x \to a} (\sin x - \sin a) = \lim_{x \to a} 2 \cdot \sin \frac{x-a}{2} \cdot \cos \frac{x+a}{2} =$

$\overset{\cancel{2}}{=} 2 \cdot \lim_{x \to a} \frac{\sin \frac{x-a}{2}}{\frac{x-a}{2}} \cdot \lim_{x \to a} \frac{x-a}{2} \cdot \lim_{x \to a} \cos \frac{x+a}{2}$

$= 2 \cdot 1 \cdot 0 \cdot \cos a = 0$

VOLSF (P) $f(y) = \frac{\sin y}{y}$

$g(x) = \frac{x-a}{2}$ $g(x) \neq 0$ na $P(a,1)$

□

Def. nechť $\sin^* x = \sin x$ pro $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

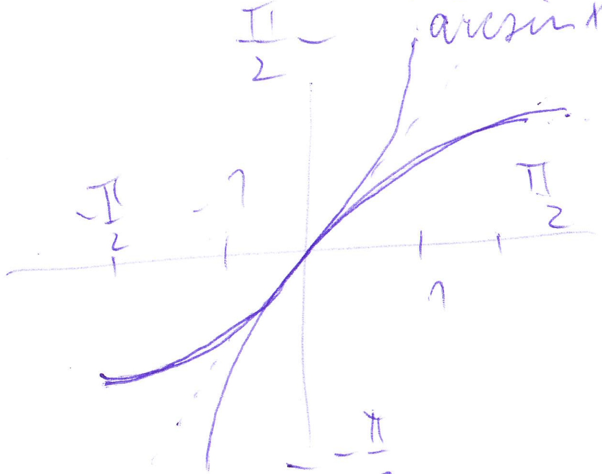
$\cos^* x = \cos x$ pro $x \in [0, \pi]$

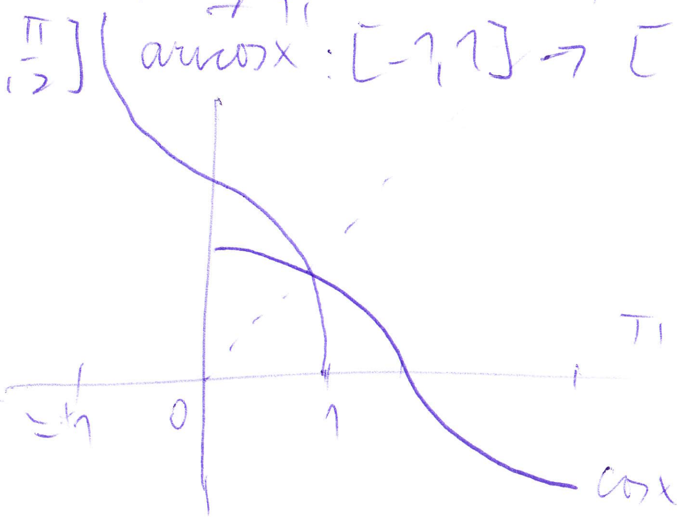
$\tan^* x = \tan x$ pro $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\cotg^* x = \cotg x$ pro $x \in (0, \pi)$

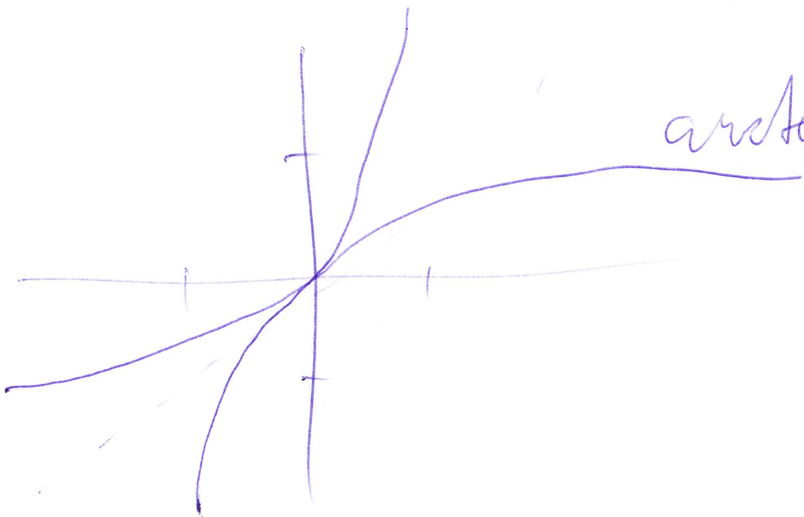
Definujeme arcsin (respektive arccos, arctan, arccotg) jako inverzní funkce k $\sin^*$ (respektive $\cos^*$, $\tan^*$, $\cotg^*$)

$\arcsin x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ | $\arccos x : [-1, 1] \to [0, \pi]$





$\arctan x : \mathbb{R} \to (-\frac{\pi}{2}, \frac{\pi}{2})$



$\mathbb{R} \to (0, \pi)$