$\boxed{\text{Věta T 3.12}}$ (zavedení exponenciely)

Existuje funkce $\exp: \mathbb{R} \to \mathbb{R}$ splňující:

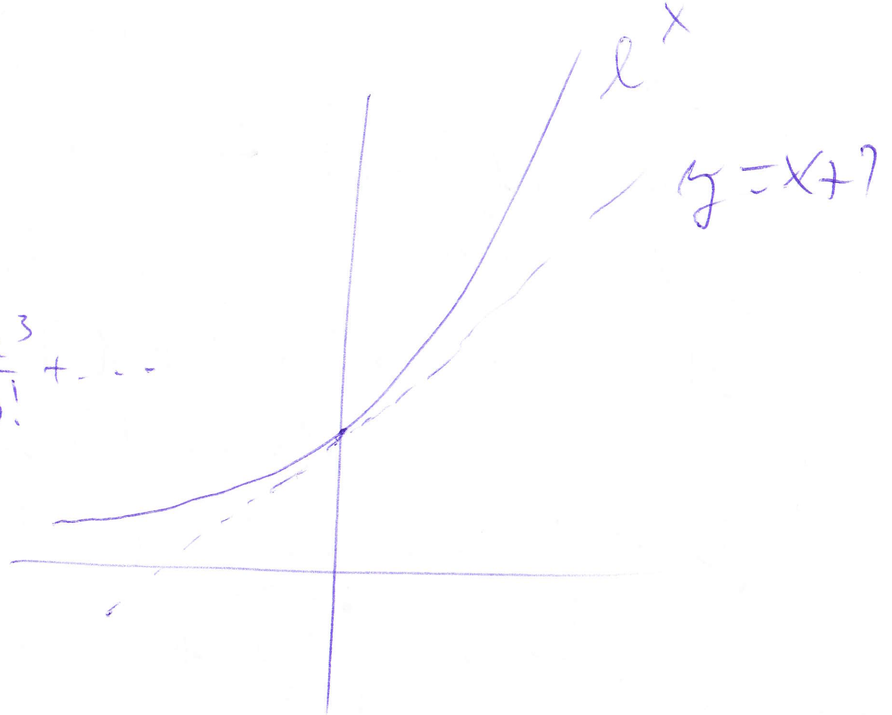
a) $\exp(x)$ je rostoucí na $\mathbb{R}$

b) $\forall x, y \in \mathbb{R}$ $\quad \exp(x+y) = \exp(x) \cdot \exp(y)$

c) $\exp(0) = 1$

d) $\lim\limits_{x \to 0} \dfrac{\exp(x) - 1}{x} = 1$

e) $\exp(x)$ je spojitá na $\mathbb{R}$.



Dk. Položme

$$\exp x = \lim_{k \to \infty} \sum_{n=0}^{k} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= \frac{1}{1} + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

c)

$$\exp 0 = \lim_{k \to \infty} \sum_{n=0}^{k} \frac{0^n}{n!} =$$

$$= \lim_{k \to \infty} 1 = 1$$

Chceme ukázat $\exists \lim\limits_{k \to \infty} \sum\limits_{n=0}^{k} \frac{x^n}{n!} \in \mathbb{R}$ pro každé $x \in \mathbb{R}$.

$$\exp x = \lim_{k\to\infty} \sum_{n=0}^{k} \frac{x^n}{n!}$$

- exp je dobře definována – lim existuje a $\in \mathbb{R}$

IDEA: $\exists \lim_{k\to\infty} \sum_{n=2}^{k} \frac{1}{n\cdot(n-1)} = \lim_{k\to\infty} 1 - \frac{1}{k} = 1$

$$\sum_{n=2}^{k} \left(\frac{1}{n-1} - \frac{1}{n}\right) = \left(1-\cancel{\frac{1}{2}}\right) + \left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}}-\frac{1}{4}\right) + \dots \left(\frac{1}{k-2}-\cancel{\frac{1}{k-1}}\right) + \left(\cancel{\frac{1}{k-1}} - \frac{1}{k}\right)$$

Nechť x je pevné, pak $\exists k_0 \quad k_0 > |x| \quad (k_0 = [|x|]+1)$

$\forall n \geq 2k_0$ platí $\left|\frac{x^n}{n!}\right| = |x|^{k_0} \underbrace{\frac{1}{k_0!}}_{\leq 1} \underbrace{\frac{|x|}{k_0+1}}_{\leq 1} \underbrace{\frac{|x|}{k_0+2}}_{\leq 1} \cdots \underbrace{\frac{|x|}{n-2}}_{\leq 1} \frac{|x|}{n-1} \frac{|x|}{n} \leq |x|^{k_0} \frac{|x|^2}{(n-1)\cdot(n)}$

IDEA: $\left|\sum_{0}^{\infty} \frac{x^n}{n!}\right| \leq \sum_{0}^{k_0} \frac{x^n}{n!} + \sum_{k_0+1}^{\infty} |x|^{k_0+2} \frac{1}{(n-1)\cdot n} < +\infty \quad x, k_0$ pevné

Podle V 2.14 (Bolzano–Cauchyova podmínka pro posloupnost) stačí ověřit BC podm.

$\wedge$ x je pevné. Nechť $\varepsilon > 0$, zvolme $n_0 \geq 2k_0$, kde $k_0 = [|x|]+1$ $\boxed{\text{a } \frac{1}{n_0} < \varepsilon.}$

Nechť $m, k \geq n_0$ BÚNO $m > k$:

$$|a_m - a_k| \quad \left|\sum_{n=0}^{m} \frac{x^n}{n!} - \sum_{n=0}^{k} \frac{x^n}{n!}\right| = \left|\sum_{n=k+1}^{m} \frac{x^n}{n!}\right| \leq \sum_{n=k+1}^{m} \left|\frac{x^n}{n!}\right| \leq$$

$$\leq \sum_{n=k+1}^{m} |x|^{k_0+2} \cdot \frac{1}{n\cdot(n-1)} = |x|^{k_0+2} \left(\frac{1}{k} - \frac{1}{k+1} + \frac{1}{k+1} - \frac{1}{k+2} + \right.$$

V 2.14 $\exists \lim a_n \in \mathbb{R} \Leftrightarrow$
$\forall \varepsilon > 0 \, \exists n_0 \, \forall m, k \geq n_0 \; |a_m - a_k| < \varepsilon$

• b) $\exp(x+y) = \exp x \cdot \exp y$

$$\frac{k!}{j!\cdot(k-j)!}$$

MYŠLENKA:

$$\exp(x+y) = \sum_{k=0}^{\infty} \frac{1}{k!}\cdot(x+y)^k \overset{\text{binom. věta}}{=} \sum_{k=0}^{\infty} \frac{1}{k!}\cdot\sum_{j=0}^{k}\binom{k}{j}\cdot x^{k-j}\cdot y^j =$$

$$= \sum_{k=0}^{\infty}\sum_{j=0}^{k}\frac{x^{k-j}}{(k-j)!}\cdot\frac{y^j}{j!} = \underbrace{\left(\sum_{i=0}^{\infty}\frac{x^i}{i!}\right)}_{\exp x}\cdot\underbrace{\left(\sum_{j=0}^{\infty}\frac{y^j}{j!}\right)}_{\exp y}$$

$$a_0 b_0 + (a_1 b_0 + a_0 b_1) + (a_2 b_0 + a_1 b_1 + a_0 b_2) + \dots = (a_0 + a_1 + a_2 + a_3 + \dots)(b_0 + b_1 + b_2 + b_3 + \dots)$$

$$= |x|^{k_0+2}\cdot\left(\left(\frac{1}{k}-\cancel{\frac{1}{k+1}}\right)+\left(\cancel{\frac{1}{k+1}}-\cancel{\frac{1}{k+2}}\right)+\cancel{\dots}+\left(\cancel{\frac{1}{m-1}}-\frac{1}{m}\right)\right)$$

$$= |x|^{k_0+2}\cdot\left(\frac{1}{k}-\frac{1}{m}\right) \le |x|^{k_0+2}\,\frac{1}{n_0} < |x|^{k_0+2}\cdot\varepsilon.$$

$$|x|^{k_0+2} = |x|^{[|x|]+3}$$

Tedy podle V 2.74 skutečně $\exists \lim_{k\to\infty}\sum_{n=0}^{k}\frac{x^n}{n!}$.

Tedy exp je dobře definována.

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- b) $\exp(x+y) = \exp x \cdot \exp y$

Označme $s_n = \sum_{i=0}^{n} \frac{1}{i!} \cdot x^i \xrightarrow{n\to\infty} s = \exp x$

$\sigma_n = \sum_{j=0}^{n} \frac{1}{j!} \cdot y^j \xrightarrow{n\to\infty} \sigma = \exp y$

$$\rho_n = \sum_{k=0}^{n} \frac{(x+y)^k}{k!} = \sum_{k=0}^{n} \sum_{j=0}^{k} \frac{x^{k-j}}{(k-j)!} \cdot \frac{y^j}{j!} \xrightarrow{n\to\infty} \rho = \exp(x+y).$$

Nechť $\varepsilon > 0$. Pak $\exists n_0$, že $\sum_{i=n_0}^{\infty} \frac{|x|^i}{i!} < \varepsilon$, $\sum_{j=n_0}^{\infty} \frac{|y|^j}{j!} < \varepsilon$

a zároveň $|s_{n_0} \cdot \sigma_{n_0} - s \cdot \sigma| < \varepsilon$.

Nechť $n \geq 2n_0$, pak

$$|\rho_n - s\cdot\sigma| \leq |\rho_n - s_{n_0} \cdot \sigma_{n_0}| + |s_{n_0} \cdot \sigma_{n_0} - s \cdot \sigma|$$

$$\leq \sum_{j=0}^{\infty} \sum_{i=n_0}^{\infty} \frac{|x|^j}{j!} \cdot \frac{|y|^i}{i!} + \sum_{j=n_0}^{\infty} \sum_{i=0}^{\infty} \frac{|x|^j}{j!} \cdot \frac{|y|^i}{i!} + \varepsilon$$

členy které se neodečtou mají alespoň jeden index $\geq n_0$, jinak se vyskytují i v $s_{n_0} \sigma_{n_0}$

$$\leq \sum_{j=0}^{\infty} \frac{|x|^j}{j!} \cdot \sum_{i=n_0}^{\infty} \frac{|y|^i}{i!} + \sum_{j=n_0}^{\infty} \frac{|x|^j}{j!} \cdot \sum_{i=0}^{\infty} \frac{|y|^i}{i!} + \varepsilon < e^{|x|} \cdot \varepsilon + \varepsilon e^{|y|} + \varepsilon$$

Tedy $\rho_n \to s \cdot \sigma = \exp x \cdot \exp y$

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a) $0 \le x \le y \Rightarrow \frac{x^n}{n!} \le \frac{y^n}{n!} \Rightarrow \sum_{n=0}^{k} \frac{x^n}{n!} \le \sum_{n=0}^{k} \frac{y^n}{n!}$

$\Rightarrow \exp x \le \exp y.$

b) pro $x < 0$ $\quad \exp x = \frac{1}{\exp(-x)}$

$x \le y \le 0 \Rightarrow -x \ge -y \ge 0 \Rightarrow \exp(-x) \ge \exp(-y) \Rightarrow$

$\exp x = \frac{1}{\exp(-x)} \le \frac{1}{\exp(-y)} = \exp y$

$x \le 0 \le y \quad \exp x \le \exp(0) \le \exp y$

$$\boxed{\begin{aligned} \exp x &= 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \\ \frac{\exp x - 1}{x} &= 1 + \frac{x}{2} + \frac{x^2}{3!} + \dots \end{aligned}}$$

d) $\lim_{x \to 0} \frac{\exp x - 1}{x} = \lim_{x \to 0} \frac{\sum_{n=1}^{\infty} \frac{x^n}{n!}}{x} =$

$= \lim_{x \to 0} 1 + x \cdot \sum_{n=2}^{\infty} \frac{x^{n-2}}{n!} = 1$

$\left| x \cdot \sum_{n=2}^{\infty} \frac{x^{n-2}}{n!} \right| \overset{|x| \le 1}{\le} |x| \cdot \sum_{n=2}^{\infty} \frac{1}{n!} \le |x| \cdot e \xrightarrow{x \to 0} 0$

e) exp je spojitá na $\mathbb{R}$

Chceme $\lim\limits_{h\to 0} \exp(x+h) = \exp(x)$ ... spojitost v x.

$$\lim_{h\to 0} \underbrace{\exp(x+h) - \exp(x)} + \exp(x) =$$

$$= \lim_{h\to 0} \frac{\exp(x)\cdot\exp(h) - \exp(x)}{h}\cdot h + \exp(x) =$$

$$= \lim_{h\to 0} \exp(x)\cdot \underbrace{\frac{\exp(h)-1}{h}}_{\xrightarrow{d)} 1} \overset{\to 0}{h} + \exp(x) = \exp x\cdot 1\cdot 0 + \exp x = \exp x. \qquad \square$$

<u>Příklad</u>: $e = \sum\limits_{n=0}^{\infty} \frac{1}{n!}$ je iracionální.

Nechť pro spor $e = \frac{p}{q}$, $p, q \in \mathbb{N}$.

$$\sum_{n=0}^{q} \frac{1}{n!} < e = \frac{p}{q} = \sum_{n=0}^{q} \frac{1}{n!} + \underline{\sum_{n=q+1}^{\infty} \frac{1}{n!}} \qquad \frac{1}{q!\cdot q}$$

$$\sum_{n=q+1}^{\infty} \frac{1}{n!} \leq \sum_{n=q+1}^{\infty} \frac{1}{(q+1)!}\cdot\frac{1}{(q+1)^{n-(q+1)}} = \frac{1}{(q+1)!}\sum_{j=0}^{\infty}\frac{1}{(q+1)^j} = \frac{1}{(q+1)!}\cdot\frac{1}{1-\frac{1}{q+1}}$$

$$\sum_{n=0}^{q} \frac{1}{n!} < e = \frac{p}{q} < \sum_{n=0}^{q} \frac{1}{n!} + \frac{1}{q!\cdot q} \qquad / \cdot q!\cdot q$$

$$\mathbb{N} \ni m < p\cdot q! < m+1 \qquad \lightning$$