$\boxed{\text{Př.}}$ Z Beekmanovy formule odvoďte vyjádření Laplaceovy transformace pravd. ruinování.

RUIN $\iff L > u$

Celková ztráta: $L = L_1 + L_2 + \dots + L_N$ (náhodný počet)

$L \geq 0$

L_i i.i.d.:

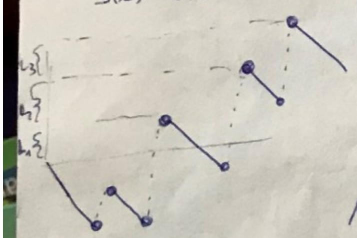
o) ... i-té překročení posl. nejvyšší hodnoty

$\left(L = \max_{t \geq 0} \left(S(t) - ct \right) \right)$ d.f. X, $\mu_1 := \mathbb{E}X$ (hustota?)

o) ... rozdělení $\dfrac{1 - P(y)}{\mu_1}$ s momentovou vytvořující

funkcí $M_{L_1}(r) = \dfrac{M_X(r) - 1}{\mu_1 \cdot r}$

$S(t) - ct$:



N ... geometrické rozdělení (nulová ztráta = neúspěch (!)) s p. $\psi(0) = P(L > 0) = \dfrac{1}{1+\theta}$, θ ... bezpečnostní přirážka

$P(N = n) = \psi(0)^n \cdot (1 - \psi(0)) = \theta \left(\dfrac{1}{1+\theta} \right)^{n+1}$, $n = 0, 1, \dots$

$P(L \leq u) =$
$= P(S(t) - ct \leq u, t \geq 0)$ (celé trajektorie !!!)
$= 1 - \psi(u)$

$\psi(u)$: $P(S(T) - cT > 0)$, $T < \infty$:

B.F.:

$\displaystyle \sum_{n=1}^{\infty} P(N=n) \cdot P\Big(\sum_i^n L_i > u\Big)$; $\boxed{P\Big(\sum_i^0 L_i > u\Big) = 0}$; pro $n = 0$ $= 1$; $\boxed{u = 0}$?

$$\psi(u) = P(L > u) = \sum_{n=0\,(1)}^{\infty} \theta \left(\frac{1}{1+\theta} \right)^{n+1} \int_u^{\infty} \left[\frac{1 - P(y)}{\mu_1} \right]^{*n} dy$$

$\uparrow$ n-násobná konvoluce

$$= \sum_{n=1}^{\infty} \frac{\theta}{1+\theta} \cdot \left(\frac{1}{1+\theta} \right)^n \cdot \int \cdot \, dy$$

$$\mathcal{L}(\psi) = \psi^*(s) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta} \right)^n \cdot \frac{\theta}{1+\theta} \cdot \left(\frac{1}{s} - \underbrace{\mathcal{L}\left(\int_0^u \left(\frac{1 - P(y)}{\mu_1} \right)^{*n} dy \right)}_{\circledast} \right)$$

$\mathcal{L}2)$, $\mathcal{L}4)$

$$\circledast \overset{\downarrow}{=} \frac{1}{s} \cdot \left(\mathcal{L}\left(\frac{1 - P(t)}{\mu_1} \right) \right)^n, \quad P(t) = \int_0^t p(x)\, dx$$

$$\mathcal{L}\left(\frac{1 - P(t)}{\mu_1} \right) = \frac{1}{\mu_1} \cdot \left(\frac{1}{s} - \frac{1}{s} \cdot \underbrace{\int_0^{\infty} e^{-st} p(t)\, dt}_{M_X(-s)} \right)$$

$\uparrow$ $\mathcal{L}1)$ & $\mathcal{L}(2)$; $\mathcal{L}(1)$

• Laplace integrálu $\overset{?}{=} \frac{1}{s} \mathcal{L}(\text{hustoty})$ $\Longrightarrow$

$$\psi^*(s) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \cdot$$

$$\cdot \left[\frac{1}{s} - \frac{1}{s}\cdot\left(\frac{1}{p_1}\cdot\left(\frac{1}{s} - \frac{1}{s} M_X(-s)\right)\right)^n\right] =$$

$$= \sum -''- \quad \frac{1}{s}\cdot\left[1-\left(\frac{1-M_X(-s)}{s p_1}\right)^n\right] =$$

$$= \frac{\theta}{1+\theta}\cdot\frac{1}{s}\cdot\frac{1}{1+\theta}\cdot\left(\frac{1}{1-\frac{1}{1+\theta}}\right) - \frac{\theta}{1+\theta}\cdot\frac{1}{s}\cdot\frac{\frac{1-M_X(-s)}{s p_1}\cdot\frac{1}{1+\theta}}{1-\frac{1-M_X(-s)}{s p_1}\cdot\frac{1}{1+\theta}} =$$

$$= \frac{1}{s\cdot(1+\theta)} - \frac{\theta}{s\cdot(1+\theta)}\cdot\frac{\cancel{(1+\theta)sp_1} + 1 - M_X(-s)}{(1+\theta)sp_1 - 1 + M_X(-s)}$$

$$\psi(0) = \frac{1}{1+\theta} \Rightarrow \mathcal{L}(\psi') = s\,\mathcal{L}(\psi) - \psi(0)$$

$\boxed{\text{Př.}}$ Stanovte pravděpodobnost ruinování pro $p(x) = a e^{-ax}$ $\quad a > 0$

$$M_X(-s) = \int_0^\infty e^{-sx}\, a\cdot e^{-ax}\, dx = a\cdot\left[\frac{e^{-x\cdot(s+a)}}{-(s+a)}\right]_0^\infty =$$

$$= a\cdot\frac{1}{s+a}$$

exponenciální rozdělení

$$EX = \frac{1}{a}$$

$$\mathcal{L}(\psi')(s) = \frac{\theta}{1+\theta}\cdot\frac{1-M_X(-s)}{1-(1+\theta)p_1 s - M_X(-s)}$$

$$= \frac{\theta}{1+\theta}\cdot\frac{1-\frac{a}{s+a}}{1-(1+\theta)\cdot\frac{s}{a}-\frac{a}{s+a}} =$$

$as + a^2$
$-(1+\theta)\cdot(s^2+sa)$
$-a^2$
$\cancel{(1+\theta)s\cdot(s+a)}$

$$= \frac{\theta}{1+\theta}\cdot\frac{s}{s+a} \Big/ \frac{(as + s^2) - \theta s^2 + a^2 - as - as\theta - a^2}{(s+a)\cdot a} =$$

$$= \frac{\theta}{1+\theta} \cdot \frac{-as}{-as + s^2 \cdot (1+\theta) + as(1+\theta)} =$$

$$= \frac{\theta}{1+\theta} \cdot \frac{-\frac{a}{1+\theta}}{\frac{\theta a}{1+\theta} + s} = \frac{-a \cdot \theta}{(1+\theta)^2} \cdot \frac{1}{s + \frac{a\theta}{1+\theta}}$$

$$\Rightarrow \quad \psi'(u) = \frac{-a\theta}{(1+\theta)^2} \cdot e^{-\frac{a\theta}{1+\theta} \cdot u}$$

$1 - P(L \le u) = P(L > u) = \quad \frac{\partial -''-}{\partial u} = -f_L = \psi'$

$$\mathcal{L}(f) = \frac{1}{s-a} \Rightarrow f = e^{at}$$

$$\psi(u) = -\int_u^\infty \psi'(y)\, dy = \int_u^\infty \frac{a\theta}{(1+\theta)^2} \cdot e^{-\frac{a\theta}{1+\theta} \cdot y}\, dy =$$

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$$= \frac{a\theta}{(1+\theta)^2} \cdot \left[\frac{e^{-\frac{a\theta y}{1+\theta}}}{\frac{-a\theta}{1+\theta}} \right]_u^\infty =$$

$$= \frac{1}{1+\theta} \cdot e^{-\frac{a\theta u}{1+\theta}}$$