$x = 0{,}5^{-1/\alpha}$

$\frac{\eta_s}{\eta_\varepsilon} x^2 - \dots 1 - \frac{\eta_s}{\eta_\varepsilon} = 0$

$x_{1,2} = \frac{1 \pm \sqrt{\dots \frac{\eta_s}{\eta_\varepsilon}\left(1 - \frac{\eta_s}{\eta_\varepsilon}\right)}}{2 \cdot \frac{\eta_s}{\eta_\varepsilon}} = \frac{1 \pm \left(2 \cdot \frac{\eta_s}{\eta_\varepsilon} - 1\right)}{2 \cdot \frac{\eta_s}{\eta_\varepsilon}} =$

$= \begin{cases} 1 \\ \frac{\eta_\varepsilon}{\eta_s} - 1 \end{cases}$

$0, \dots 1 \Rightarrow \alpha = \infty$

$0{,}5^{-1/\alpha} = \dots 1$

$-\frac{1}{\alpha} \log 0{,}5 = \dots \left(\frac{\eta_\varepsilon}{\eta_s} - 1\right)$

$\alpha = \frac{-\log \dots}{\log\left(\frac{\eta_\varepsilon}{\eta_s} - 1\right)}$

Model kolektivního rizika

Adjustační koeficient R: $Y_n = X_n - c \cdot \tau_n$, $c = \lambda(1+\theta)p_1$

$$M_Y(R) = 1$$

$$1 = \frac{\lambda}{\lambda + cR} M_X(R)$$

$$1 + (1+\theta)p_1 R = M_X(R)$$

Př. Určete adj. koef. R, je-li rozdělení X dáno hustotou

$$p(x) = \frac{\sqrt{\beta}}{\Gamma(\frac{1}{2})} x^{-1/2} e^{-\beta x}, \quad x > 0 \qquad X \sim \Gamma(\tfrac{1}{2}, \beta)$$

$\Gamma(p-1) = \int_0^\infty$

$$EX = \int_0^\infty x \frac{\sqrt{\beta}}{\Gamma(\frac{1}{2})} x^{-\frac{1}{2}} e^{-\beta x} \, dx = \int_0^\infty \left(\frac{y}{\beta}\right)^{1/2} \frac{\beta^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} e^{-y} \frac{1}{\beta} \, dy =$$

$$= \frac{1}{\Gamma(\frac{1}{2})} \cdot \frac{1}{\beta} \int_0^\infty y^{1/2} e^{-y} \, dy = \frac{1}{2\beta}$$

$$M_X(R) = E e^{RX} = \int_0^\infty e^{Rx} \cdot \frac{\sqrt{\beta}}{\Gamma(\frac{1}{2})} x^{-1/2} e^{-\beta x} \, dx = \int_0^\infty \frac{\beta^{1/2}}{\Gamma(\frac{1}{2})} \left(\frac{y}{\beta - R}\right)^{-1/2} e^{-y} \frac{1}{\beta - R} \, dy =$$

$$= \frac{\sqrt{(\beta - R)\beta}}{\Gamma(\frac{1}{2})(\beta - R)} \cdot \Gamma(\tfrac{1}{2}) = \sqrt{\frac{\beta}{\beta - R}}$$

$$1+(1+\theta)\frac{1}{2\beta}R=\frac{\sqrt{\beta}}{\sqrt{\beta-R}}$$

$R=\beta-x^2$

$x=\sqrt{\beta-R}$

$$1+aR=\frac{\sqrt{\beta}}{x}\Rightarrow x+aRx=\sqrt{\beta}\Rightarrow x+a\beta x-ax^3=\sqrt{\beta}$$

$$ax^3-(1+a\beta)x+\sqrt{\beta}=0$$

$R=0$

$$(ax^3-(1+a\beta)x+\sqrt{\beta}):(x-\sqrt{\beta})=ax^2+a\sqrt{\beta}x-1$$

$$-(ax^3-a\sqrt{\beta}x^2)$$

$$a\sqrt{\beta}x^2-(1+a\beta)x+\sqrt{\beta}$$

$$-(a\sqrt{\beta}x^2-a\beta x)$$

$$-x+\sqrt{\beta}$$

$$ax^2+a\sqrt{\beta}x-1=0 \qquad a=(1+\theta)\frac{1}{2\beta}$$

$$x_{1,2}=\frac{-a\sqrt{\beta}\pm\sqrt{a^2\beta+4a}}{2a}=\frac{-\sqrt{\beta}\pm\sqrt{\beta+\frac{4}{a}}}{2}$$

pro + je R<0

$$R=\beta-x^2=\beta-\left(\sqrt{\frac{\beta}{4}+\frac{1}{a}}\mp\frac{\sqrt{\beta}}{2}\right)^2$$

$$R=\beta\cdot\left(1-\left(\sqrt{\frac{1}{4}+\frac{2}{1+\theta}}-\frac{1}{2}\right)^2\right)>0$$

Př: Z Beekmanovy formule odvoďte vyjádření Laplaceovy transformace pravděpodobnosti ruinování

B.F. $\psi(u)=P(L_1+\ldots+L_N>u)=\sum_{n=1}^{\infty}P(N=n)P(L_1+\ldots+L_n>u)$

$\psi(0)=\frac{1}{1+\theta}$ $\qquad L_i$ ... i-té překročení poslední největší hodnoty

L_i ... hustota $\frac{1-P(y)}{p_1}$, $\quad P(y)$ d.f. X

$p_1=EX$

$$\psi(u) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \int_u^{\infty} \left(\frac{1-P(y)}{p_1}\right)^{*n} dy$$

$$\mathcal{L}(\psi) = \psi^*(s) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \left(\frac{1}{s} - \mathcal{L}\left(\int_0^u \left(\frac{1-P(y)}{p_1}\right)^{*n} dy\right)\right) =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \left(\frac{1}{s} - \frac{1}{s} \cdot \left(\mathcal{L}\left(\frac{1-P(u)}{p_1}\right)\right)^n\right)$$

$$P(y) = \int_0^y p(x)\,dx$$

$$\mathcal{L}\left(\frac{1-P(y)}{p_1}\right) = \frac{1}{p_1}\left(\frac{1}{s} - \frac{1}{s} \underbrace{\int_0^{\infty} e^{-st} p(t)\,dt}_{M_X(-s)}\right)$$

$$\psi^*(u) = \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \left[\frac{1}{s} - \frac{1}{s} \cdot \left(\frac{1}{p_1}\left(\frac{1}{s} - \frac{1}{s} M_X(-s)\right)\right)^n\right] =$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)^n \frac{\theta}{1+\theta} \frac{1}{s} \left[1 - \left(\frac{1-M_X(-s)}{s p_1}\right)^n\right] =$$

$$= \cancel{\sum_{n=1}^{\infty} \left(1 \right.}$$

$$= \cancel{\frac{\theta}{1+\theta} \frac{1}{s} \sum_{n=1}^{\infty} \left(\frac{1}{1+\theta}\right)}$$

$$\frac{\theta}{1+\theta} \frac{1}{s} \cdot \frac{1}{(1+\theta)} \cdot \frac{1}{1 - \frac{1}{1+\theta}} - \frac{\theta}{1+\theta} \frac{1}{s} \cdot \frac{\frac{1-M_X(-s)}{s p_1}}{1 - \frac{1-M_X(-s)}{s p_1}} =$$

$$= \frac{1}{s(1+\theta)} - \frac{\theta}{s(1+\theta)} \cdot \frac{1-M_X(-s)}{s p_1 - 1 + M_X(-s)}$$

$$\psi(0) = \frac{1}{1+\theta}$$

$$\mathcal{L}(\psi') = s\mathcal{L}(\psi) - \psi(0)$$

$$\mathcal{L}(\psi') = \frac{\theta}{1+\theta} \frac{1-M_X(-s)}{1 - (1+\theta) p_1 s - M_X(-s)}$$

Př: Stanovte pravděpodobnost ruinování pro $p(x) = a e^{-ax}$, $a>0$

$$M_X(-s) = \int_0^{\infty} e^{-st} \cdot a \cdot e^{-at}\,dt = a \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^{\infty} = a \frac{1}{s+a}$$

$$EX = \frac{1}{a}$$

$$\mathcal{L}(\psi') = \frac{\theta}{1+\theta} \cdot \frac{1 - \frac{a}{s+a}}{1 + \frac{(1+\theta)}{a} \cdot s - \frac{a}{s+a}} = \frac{\theta}{1+\theta} \frac{\frac{s}{s+a}}{\frac{as + s^2 + \theta s^2 + a^2 + as + as\theta - a^2}{(s+a)a}} =$$

$$= \frac{\theta}{1+\theta} \frac{as}{as + s^2(1+\theta) + as(1+\theta)} = \frac{a}{a + s(1+\theta) + a(1+\theta)} \cdot \frac{\theta}{1+\theta} =$$

$$\mathcal{L}(t^n e^{qt}) = \frac{\Gamma(n+1)}{(s-q)^{n+1}} \qquad \mathcal{L}(f) = \frac{1}{s-q} \Rightarrow f = e^{qt}$$

$$= \frac{\theta}{1+\theta} \cdot \frac{\frac{-a}{1+\theta}}{\frac{\theta a}{1+\theta} + s} = \frac{-a\theta}{(1+\theta)^2} \cdot \frac{1}{s + \frac{a\theta}{1+\theta}}$$

$$\psi'(u) = \frac{-a\theta}{(1+\theta)^2} e^{-\frac{a\theta}{1+\theta} u}$$

$$\underline{\underline{\psi(u)}} = \int_u^\infty -\psi'(y)\,dy = \int_u^\infty \frac{a\theta}{(1+\theta)^2} e^{-\frac{a\theta}{1+\theta} y}\,dy =$$

$$= \frac{a\theta}{(1+\theta)^2} \cdot \left[\frac{e^{-\frac{a\theta}{1+\theta} y}}{\frac{-a\theta}{1+\theta}} \right]_u^\infty = \underline{\underline{\frac{1}{1+\theta} e^{\frac{-a\theta u}{1+\theta}}}}$$

Př. K odvození $\mathcal{L}(\psi)$ lze použít místo Bellman. formule rovnici uvedenou v přednášce na základě úvahy o velikosti prvního překročení nuly celkovou ztrátou.

$$\psi(u) = \underbrace{\frac{1}{1+\theta}}_{\text{překročení nastane}} \underbrace{\int_u^\infty \frac{1-P(y)}{p_1}\,dy}_{P(L>u)} + \frac{1}{1+\theta} \int_0^u \frac{1-P(y)}{p_1} \psi(u-y)\,dy$$

$$\psi^*(s) = \frac{1}{1+\theta}\left(\frac{1}{s} - \frac{1}{s} \frac{1-M_X(-s)}{p_1 s}\right) + \frac{1}{1+\theta}\left(\frac{1-M_X(-s)}{p_1 s} \psi^*(s)\right)$$

$$\psi^*(s)\left[1 - \frac{1}{1+\theta} \frac{1-M_X(-s)}{p_1 s}\right] = \frac{1}{1+\theta} \frac{1}{s}\left[1 - \frac{1-M_X(-s)}{p_1 s}\right]$$

$$\psi^*(s) = \frac{1}{s(1+\theta)} - \frac{1}{(1+\theta)s} \frac{\theta(1-M_X(-s))}{(1+\theta)p_1 s - 1 + M_X(-s)}$$

Vysoké škody

Př. 1 Ověřte, že rozdělení s hustotou $p(x) = \frac{a}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-\frac{a^2}{2x}}$ je subexponenciálního typu

d.f. $P(x) = 2\left(1 - \Phi\left(\frac{a}{\sqrt{x}}\right)\right)$

$\not\exists\, (t)$ pro $t \in (0,\infty)$ musí ex. $\Rightarrow$ subexponenciální

$$\gamma(t) = \lim_{x\to\infty} \frac{1-P(tx)}{1-P(x)} \overset{l'H}{=} \lim_{x\to\infty} \frac{t\cdot p(tx)}{p(x)} = \lim_{x\to\infty} \frac{t\cdot\frac{a}{\sqrt{2\pi}}\, t^{-3/2}\cdot x^{-3/2} e^{-\frac{a^2}{2tx}}}{\frac{a}{\sqrt{2\pi}}\, x^{-3/2} e^{-\frac{a^2}{2x}}} =$$

$$= \lim_{x\to\infty} \frac{t^{-1/2}\, e^{-\frac{a^2}{2tx}+\frac{a^2}{2x}}}{1} = \frac{1}{\sqrt{t}} \quad , \; t\in(0,1)$$

Př 2: Log.normální rozdělení je subexp. typu.

Stanovte $\gamma(t)$, $t\in(0,1)$

$\xi \sim N(\mu,\sigma^2)$ $\qquad P(e^\xi \le x) = P(\xi\le \log x) = \Phi\left(\frac{\log x-\mu}{\sigma}\right)$

$$\gamma(t) = \lim_{x\to\infty} \frac{1-\Phi\left(\frac{\log tx-\mu}{\sigma}\right)}{1-\Phi\left(\frac{\log x-\mu}{\sigma}\right)} \overset{l'H}{=}$$

$$= \lim_{x\to\infty} \frac{-\varphi\left(\frac{\log tx-\mu}{\sigma}\right)\cdot\frac{1\cdot t}{\sigma t x}}{-\varphi\left(\frac{\log x-\mu}{\sigma}\right)\frac{1}{\sigma x}} = \lim_{x\to\infty} \frac{e^{-\frac{(\log tx-\mu)^2}{2\sigma^2}}}{e^{-\frac{(\log x-\mu)^2}{2\sigma^2}}} =$$

$$= \lim_{x\to\infty} e^{-\frac{1}{2\sigma^2}\cdot(\log^2 tx - 2\log tx\cdot\mu+\mu^2-\log^2 x+2\log x\,\mu-\mu^2)} =$$

$$= \lim_{x\to\infty} \exp\left[-\frac{1}{2\sigma^2}\left(\log^2 t+2\log t\log x+\cancel{\log^2 x}-\cancel{2\log x\mu}-2\log t-\cancel{\log^2 x}+\cancel{2\log x\mu}\right)\right] =$$

$$= \lim_{x\to\infty} \exp\left[-\frac{1}{2\sigma^2}\left(\log^2 t-2\log t+\underbrace{2\underbrace{\log t}_{<0}\cdot\underbrace{\log x}_{\to\infty}}_{\to-\infty}\right)\right] = \infty$$

Př 3: Pro vysoké škody platí $\psi(u)\sim\frac{1}{\theta\mu_1}\int_u^\infty(1-P(y))\,dy$, $u\to\infty$.

Odvoďte asymptotickou formuli pro $\psi(u)$, $u\to\infty$, mají-li výše škod logaritmicko-normální rozdělení.

DK: $\quad P(y) = \Phi\left(\frac{\log y-\mu}{\sigma}\right)$

$\qquad 1-\Phi(x)\sim \frac{1}{x}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}$, $x\to\infty$

$$\lim_{x\to\infty}\frac{1-\Phi(x)}{\frac{1}{x}e^{-\frac{x^2}{2}}\frac{1}{\sqrt{2\pi}}} \overset{l'H}{=} \lim_{x\to\infty}\frac{-\varphi(x)}{f'(x)} = 1$$

$$\int_u^\infty(1-P(y))\,dy \sim \frac{1}{\sqrt{2\pi}}\int_u^\infty \frac{\sigma}{\log y-\mu}\, e^{-\frac{(\log y-\mu)^2}{2\sigma^2}}\,dy$$

Hledáme fci $g(u)$, aby $\lim_{u\to\infty} \frac{\int_u^\infty f(y)\,dy}{g(u)} = 1 = \lim_{u\to\infty} \frac{-f(u)}{g'(u)}$

$$f(y) = \frac{1}{\log y}\, e^{-\frac{(\log y-\mu)^2}{2\sigma^2}}$$

$$1 = \lim_{u\to\infty} \frac{e^{\frac{-\mu^2}{2\sigma^2}} \cdot e^{-\frac{(\log u)^2}{2\sigma^2}}\ \frac{u^{\frac{\mu}{\sigma^2}}}{\log u}}{g'(u)}$$

$$g(u) = \sigma^2 \cdot e^{-\frac{\mu^2}{2\sigma^2}}\ \frac{u^{\frac{\mu}{\sigma^2}+1}}{(\log u)^2}\ e^{-\frac{(\log u)^2}{2\sigma^2}}$$

$$g'(u) = \ldots$$ dosadíme do limity, vyjde 1

$$p_1 = EX = e^{\mu+\frac{\sigma^2}{2}}$$

$$\frac{1}{p_1} = e^{-\mu-\frac{\sigma^2}{2}}$$

$$\Psi(u) \sim \frac{e^{-\mu-\frac{\sigma^2}{2}}\,\sigma}{\theta\sqrt{2\pi}}\, g(u), \quad u\to\infty$$

statistikum

Bayesův princip

neznámý parametr θ – n.v., která má pstní [illegible] dělení

$\mathbb{X} = (X_1,\ldots,X_m)^T$ [illegible] pstní hustotu $f(\ldots)$, $\theta = (\theta_1,\ldots,\theta_k)$ parametr

K závěrům o parametru θ vyjdeme [illegible] z náh. výběru a apriorní informaci. Apriorní info. [illegible] dána [illegible] tím, že θ má hustotu $g(\theta)$.

Bayes: Nechť $\theta = (\theta_1,\ldots,\theta_k)$ [illegible] hustotu $g(\theta)$. $\mathbb{X} = (X_1,\ldots,X_m)^T$ má podmíněnou hustotu [illegible] (θ) [illegible] daném θ. Pro podmíněnou hustotu $h(\theta/x)$ náh. [illegible] θ při [illegible] x platí

$$h(\theta/x) = \frac{f(x|\theta)\,g(\theta)}{\int g(\theta) f(x|\theta)\, d\lambda(\theta)}$$ [illegible] -li jmenovatel $\neq 0$, jinak 0.

$d\theta$