8. cvičení

Př 8A: $a_n := \frac{1}{n}$; $b_n := 0$ $\quad \lim_{n\to+\infty} a_n = 0 = \lim_{n\to+\infty} b_n$

$$A = B$$

$A \leq B$ ale $\forall n \in \mathbb{N}: a_n > b_n$

8B: viz výše s prohozenou rolí a_n a b_n

$\longrightarrow$ ani jedno tvrzení neplatí

9: $A \in (0,+\infty)$: $a_n = \frac{1}{n}$, $b_n = n \cdot A$

$A = 0$ $\qquad b_n = \sqrt{n}$

$A = +\infty$: $\qquad b_n = n^2$

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1. $$\lim_{n\to+\infty} \frac{\lg(n^2+n)}{\lg(n^3-1)} = \lim_{n\to+\infty} \frac{\lg\left(n^2\left[1+\frac{1}{n}\right]\right)}{\lg\left(n^3\left[1-\frac{1}{n^3}\right]\right)}$$

$$= \lim_{n\to+\infty} \frac{\lg n^{2} + \lg\left(1+\frac{1}{n}\right)}{\lg n^{3} + \lg\left(1-\frac{1}{n^3}\right)} = \lim_{n\to+\infty} \frac{\cancel{\lg n}\left(2 + \frac{\lg\left(1+\frac{1}{n}\right)}{\lg n}\right)}{\cancel{\lg n}\left(3 + \frac{\lg\left(1-\frac{1}{n^3}\right)}{\lg n}\right)}$$

$$= \frac{2}{3}, \text{ protože } \lim_{n\to+\infty} \frac{\lg\left(1+\frac{1}{n}\right)}{\lg n} = 0 \text{ a } \lim_{n\to+\infty} \frac{\lg\left(1-\frac{1}{n^3}\right)}{\lg n} = 0$$

(posl. $\left\{\lg\left(1+\frac{1}{n}\right)\right\}_{n=1}^{+\infty}$ i $\left\{\lg\left(1-\frac{1}{n^3}\right)\right\}_{n=2}^{+\infty}$ jsou omezené

a $\frac{1}{\lg n} \xrightarrow{n\to+\infty} 0$)

2. $$\lim_{n\to+\infty} \frac{\lg(n^2+3^n)}{\lg(1+n+e^{2n})} = \lim_{n\to+\infty} \frac{\lg\left(3^n\left(\frac{n^2}{3^n}+1\right)\right)}{\lg\left(e^{2n}\left(\frac{1}{e^{2n}}+\frac{n}{e^{2n}}+1\right)\right)}$$

$$= \lim_{n\to+\infty} \frac{n\lg 3 + \lg\left(1+\frac{n^2}{3^n}\right)}{n\underbrace{\lg e^2}_{=2} + \lg\left(1+\frac{1}{e^{2n}}+\frac{n}{e^{2n}}\right)} =$$

$\lg \dots$ přirozený log

$\lg e^2 = 2\lg e = 2$

$$= \lim_{n\to+\infty} \frac{\cancel{n}\left(\lg 3 + \frac{\lg\left(1+\frac{n^2}{3^n}\right)}{n}\right)}{\cancel{n}\left(2 + \frac{\lg\left(1+\frac{1}{e^{2n}}+\frac{n}{e^{2n}}\right)}{n}\right)} = \frac{\lg 3}{2}, \text{ protože}$$

$\left\{\lg\left(1+\frac{n^2}{3^n}\right)\right\}$ i $\left\{\lg\left(1+\frac{1}{e^{2n}}+\frac{n}{e^{2n}}\right)\right\}$ jsou omezené a

$\frac{1}{n} \xrightarrow{n\to+\infty} 0$ a tedy $\lim\limits_{n\to+\infty} \frac{\lg\left(1+\frac{n^2}{3^n}\right)}{n} = 0$ a

$$\lim_{n\to+\infty} \frac{\lg\left(1+\frac{1}{e^{2n}}+\frac{n}{e^{2n}}\right)}{n} = 0.$$

3. $$\lim_{n\to+\infty} \frac{\sqrt{n+\sin^2 n}-\sqrt{n-\cos^2 n}}{\sqrt{n+1}-\sqrt{n-1}} =$$

$$\lim_{n\to+\infty} \underbrace{\frac{n+\sin^2 n - n + \cos^2 n}{n+1-n+1}}_{=\frac{1}{2}} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+\sin^2 n}+\sqrt{n-\cos^2 n}} =$$

$$\lim_{n\to+\infty} \frac{1}{2} \frac{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}{\sqrt{n}\left(\sqrt{1+\frac{\sin^2 n}{n}}+\sqrt{1-\frac{\cos^2 n}{n}}\right)} = \frac{1}{2}$$

4.

$$\lim_{n\to+\infty}\sqrt[n]{\sqrt{3^n+2\cdot 2^n}-\sqrt{3^n+2^n}}=\lim_{n\to+\infty}\sqrt[n]{\frac{\cancel{3^n}+2\cdot 2^n-\cancel{3^n}-2^n}{\sqrt{3^n+2\cdot 2^n}+\sqrt{3^n+2^n}}}$$

$$=\lim_{n\to+\infty}\sqrt[n]{\frac{2^n}{\sqrt{3^n}\left(\sqrt{1+2\frac{2^n}{3^n}}+\sqrt{1+\frac{2^n}{3^n}}\right)}}=$$

$$\lim_{n\to+\infty}\frac{2}{\sqrt{3}}\sqrt[n]{\frac{1}{\sqrt{1+2\frac{2^n}{3^n}}+\sqrt{1+\frac{2^n}{3^n}}}}=\frac{2}{\sqrt{3}},\ \text{protože}$$

$$\sqrt[n]{\frac{1}{\sqrt{3}+\sqrt{2}}}\le\sqrt[n]{\frac{1}{\sqrt{1+2\frac{2^n}{3^n}}+\sqrt{1+\frac{2^n}{3^n}}}}\le\sqrt[n]{\frac{1}{2}}$$

$$\sqrt[n]{\frac{1}{\sqrt{3}+\sqrt{2}}}\xrightarrow{n\to+\infty}1,\qquad \sqrt[n]{\frac{1}{2}}\xrightarrow{n\to+\infty}1$$

5. Trik: Rozklad na parciální zlomky

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) = 1 \underbrace{- \frac{1}{2} + \frac{1}{2}}_{=0} \underbrace{- \frac{1}{3} + \frac{1}{3}}_{=0} \underbrace{\cdots \frac{1}{n}}_{=0} - \frac{1}{n+1}$$

$$= 1 - \frac{1}{n+1}$$

Tedy $\lim_{n \to +\infty} \sum_{k=1}^{n} \frac{1}{k(k+1)} = 1 + 0 = 1$

6. $1 - \frac{1}{n^2} = \frac{n^2 - 1}{n^2}$

6.

$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\cdots\left(1-\frac{1}{(n-1)^2}\right)\cdot\left(1-\frac{1}{n^2}\right)=$$

$$=\frac{(2-1)(2+1)}{2\cdot 2}\cdot\frac{(3-1)(3+1)}{3\cdot 3}\cdots\frac{(n-2)(n\cancel{+1})}{(n-1)(n-1)}\cdot\frac{(n-1)(n+1)}{n\cdot n}$$

$$=\frac{1}{2}\cdot\left(\frac{n+1}{n}\right)\xrightarrow{n\to+\infty}\frac{1}{2}$$