The Remez algorithm (The exchange alg.)

Problem: $f \in C[a,b]$, A … an $(n+1)$ dim. lin. subspace of $C[a,b]$ that satisfies the Haar condition

Look for: $p^* \in A$:

$$\|f-p^*\|_\infty = \min_{p\in A} \underbrace{\max_{x\in[a,b]} |f(x)-p(x)|}_{\|f-p\|_\infty}$$

Remark. Instead of $\|\cdot\|_\infty$ we can consider $\|\cdot\|_Z$

$Z \subseteq [a,b]$ compact, containing at least $n+1$ points

~~points~~ $n+1$ ⟹ interpolation
$n+2$ points system of lin. eq.

Remez 1934 → an iterative algorithm, it exploits ~~uses~~ characteristic properties of the best approximation, look for Chebyshev alternation set
→ adjust a reference in each step → it converges to ↑

- Given an initial reference

  1. Determine $q^* \in A$ that minimizes the maximum on the actual reference (system of lin. eq.)
  2. Test the quality of q^*
  3. Define a new reference

- Recall → minimax on the given reference

  $\{\phi_j\}_{j=0}^{n}$ … a basis of A
  $\{\xi_i\}_{i=0}^{n+1}$ … a reference

  $$q^* = \sum_{j=0}^{n} \alpha_j \phi_j$$

  α_j solve the system

$$\begin{array}{c} \xi_0 \to \\ \\ \\ \xi_{n+1} \to \end{array} \left[\begin{array}{c|c} & 1 \\ & -1 \\ A & 1 \\ & -1 \\ & \vdots \end{array}\right] \begin{bmatrix} \alpha_0 \\ \vdots \\ \alpha_n \\ h \end{bmatrix} = \begin{bmatrix} f(\xi_0) \\ \vdots \\ \\ f(\xi_{n+1}) \end{bmatrix}$$

$$\begin{array}{cc} \uparrow & \uparrow \\ \phi_0 & \phi_n \end{array}$$

$\{a_{i,j}\}_{i=1,\dots,n+2,\ j=1,\dots,n+1}$

$$a_{i,j} = \phi_{j-1}(\xi_{i-1})$$

q^* solves

$$|h| = \max_{i=0,\dots,n+1} |f(\xi_i) - q^*(\xi_i)| = \min_{p\in A} \max_i |f(\xi_i) - p(\xi_i)|$$

Vallée-Poussin:
$$\underset{\text{(points)}}{|h|} \le \underset{\substack{\text{(interval)}\\ \text{optimum}}}{\|f-p^*\|_\infty} \le \underset{\text{(interval)}}{\|f-q^*\|_\infty}$$

if q^* is not optimal → strict $<$

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- How to change (adjust) the reference to increase $|h|$?

$$h = h(\xi_0, \xi_1, \dots, \xi_{n+1})$$

- Stopping criterion. Having q^*, we can compute

$$\delta \equiv \|f-q^*\|_\infty - |h|.$$

$$\underbrace{|h| \le \|f-p^*\|_\infty \le \|f-q^*\|_\infty}_{\delta}$$

Then

not necessary to mention

$$\boxed{\|f-q^*\|_\infty = |h| + \delta \le \|f-p^*\|_\infty + \delta}$$

$\to$ stop if δ is sufficiently small.

- one-point exchange alg.

$$(\xi_0, \dots, \xi_{m+1}) \longrightarrow (\xi_0^+, \dots, \xi_{m+1}^+)$$

- adjust only 1 point s.t.

$$h(\xi_0, \dots, \xi_{m+1}) < h(\xi_0^+, \dots, \xi_{m+1}^+)$$

make it as large as possible
preserve the alternation properties

$\to$ we change only 1 row of A $\to$ update factorization

$\to$ choose a new point s.t. $|f(\eta) - q^*(\eta)| = \|f-q^*\|_\infty$

then

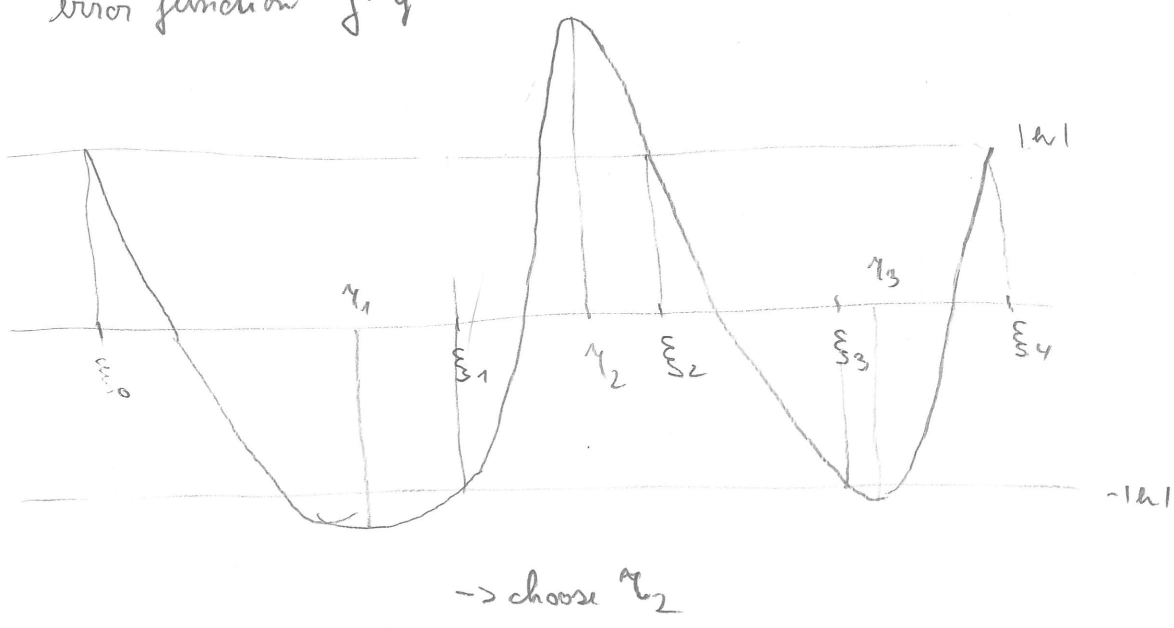
$$|h| \le \min_i |f(\xi_i^+) - q^*(\xi_i^+)| < \underbrace{\min_{p\in\mathcal{P}} \max_i |f(\xi_i^+) - p(\xi_i^+)|}_{|h(\xi_0^+, \dots, \xi_{m+1}^+)|}$$

old h

all points except 1 otherwise max.

since the values $|f(\xi_i^+) - q^*(\xi_i^+)|$ are not all equal (Vallée-Poussin) and $\delta > 0$.

error function $f - q^*$



$\to$ choose η_2

Which point leaves the reference?

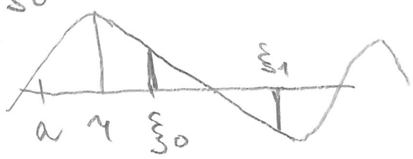
New η: $|f(\eta)-q^*(\eta)| = \|f-q^*\|_\infty$

-> preserve the alternation property

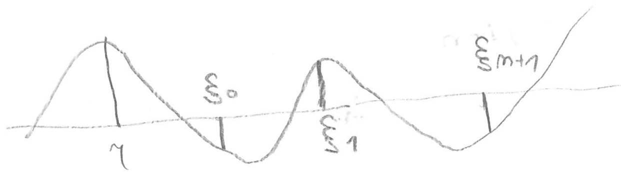
- $\eta \in (\xi_0, \xi_{m+1})$ => η lies between 2 points (ξ_i, ξ_{i+1})

  -> remove one of them s.t. $[f(\eta)-q^*(\eta)]$ and $[f(\xi_j)-q^*(\xi_j)]$ have the same sign, see figure

- $\eta < \xi_0$



or



remove either ξ_0 or ξ_{m+1}

- $\eta > \xi_{m+1}$ analogously

Initial reference -> take the Cheb. points $\xi_i = \cos\left(\frac{k\pi}{m+1}\right)$, $i=0,\dots,m+1$

-> optimal if $f \in \mathcal{P}_{m+1}$, $A = \mathcal{P}_m$

otherwise a heuristic

Special case -> minimax on a discrete set of points

$a \le x_1 < x_2 < \dots < x_m \le b$

$m > \dim A$, $\dim A = n+1$, Haar condition

- apply Remez -> each reference a subset of $\{x_i\}$.
- it has to converge (h increases strictly monotonically & finite number of different references the alg. does not go through 1 ref twice)

No presentation

~~Remark~~ Sometimes A does not satisfy the Haar condition

-> we solve the problem $\boxed{\min_{\theta \ge 0} \theta}$ ← objective function

s.t. $\boxed{-\theta \le f(x_i) - \sum_{j=0}^{m} \alpha_j \phi_j(x_i) \le \theta}$, $i = 1,\dots,m$ ← constraints

Unknown: $\alpha_0, \dots, \alpha_m, \theta$

↓

linear programming problem (optimisation)

Convergence

- Let A satisfy the Haar condition. Let $p_k \in A$ be the function calculated by the kth it. of the one-point exchange alg. Then $\{p_k\}$ converges to p^*. [Thm 9.3]

- If the functions are sufficiently smooth (twice differentiable) and some regularity properties are satisfied → quadratic convergence [Thm 9.6]

$$\exists \beta > 0 \text{ and } K \in \mathbb{N} : \forall k > K :$$

$$\| p^* - p_{k+m+2} \|_\infty \leq \beta \| p^* - p_k \|_\infty^2 .$$

p.102
- max. value of the modulus of $e^*(x)$ occurs at only $m+2$ points $\xi_0^*, \dots, \xi_{m+1}^*$
- if $\xi_0^* = a$, then $(e^*)'(a) \neq 0$
- if $\xi_{m+1}^* = b$, $(e^*)'(b) \neq 0$
- $(e^*)''(\xi_i^*) \neq 0 \ \forall i$

- Remez alg solves the optimisation problem

$$\max |h(\xi_0, \dots, \xi_{m+1})|$$

→ more efficient than the standard optim. alg.

↓ only superlinear conv

$$\frac{\| p^* - p_{k+1} \|_\infty}{\| p^* - p_k \|_\infty} \leq a_k \qquad a_k \to 0+ \quad k \to \infty$$

a_k → positive

- Present the matlab code

- The case

$$\min_{p \in \Pi_k} \max_{\lambda_i} |p(\lambda_i)|$$

↘ formulate as an approx. problem

$$\min_{q \in P_{k-1}} \max_{\lambda_i \geq 0} |1 - x q(x)|$$

↓ or $\lambda_i \neq 0$

$f \equiv 1$ ... the approx. fcion

$A = \{ x \cdot q(x) : \deg q(x) \leq k-1 \}$

k - dimensional

$x \cdot q(x)$ ... has at most k roots on $Z = \{\lambda_i\} \neq 0$

→ satisfies the Haar condition

$p \in A \Leftrightarrow p = x \sum \alpha_i x^i$