B. Polynomial interpolation

Lagrange interpolation

$\mathcal{B} = C[a,b]$, $\mathcal{A} = \mathcal{P}_n$ polynomials of degree at most n

Problem
Given $f \in C[a,b]$ and $n+1$ distinct points $x_0, x_1, \dots, x_n$ of $[a,b]$.

Look for $p \in \mathcal{P}_n$ s.t.

$$(*) \qquad p(x_i) = f(x_i), \quad i = 0, \dots, n.$$

the Lagrange interpolation pol. $\qquad x_i \dots$ interpol. points

Recall: a construction of p satisfying $(*)$

$$\boxed{l_j(x) \equiv \prod_{\substack{i=0\\ i\neq j}}^{n} \frac{x - x_i}{x_j - x_i}} \dots \text{ the } j\text{th Lagrange pol.}$$

$l_j \in \mathcal{P}_n$, $l_j(x_i) = \delta_{ij}$ $\Rightarrow$ $l_0, \dots, l_n$ a basis of $\mathcal{P}_n$

Then $p(x) = \sum_{j=0}^{n} \gamma_j l_j(x)$, γ_j determined such that $(*)$ holds.
$\Rightarrow \gamma_j = f(x_j)$

Lagrange 1795, Euler 1783

$$(**) \qquad \boxed{L_n(x) \equiv \sum_{j=0}^{n} f(x_j)\, l_j(x)} \in \mathcal{P}_n \text{ and satisfies } (*).$$

the Lagrange interpolation formula

Theorem: There is exactly one polynomial $p \in \mathcal{P}_n$ that satisfies $(*)$.

Proof. We have shown the existence, uniqueness by contradiction. ☒

- Interpolation process is an operator $X : f \in C[a,b] \longmapsto p_n(x) \in \mathcal{P}_n$

depends on the choice of x_i

From $(**)$ is is obvious is a linear projection.

X is a projection $\Rightarrow$ $\forall q \in \mathcal{P}_n$: $L_n(x) = \sum_{j=0}^{n} q(x_j)\, l_j(x)$ and $q(x)$ agree in $n+1$ points $\Rightarrow$ must be equal everywhere

In particular, for $q(x) \equiv 1$ we get

$$(+) \qquad \boxed{1 \equiv \sum_{j=0}^{n} l_j(x)}$$

- Computationally $\rightarrow$ to evaluate $L_n(x)$ using $(**)$ we need $O(n^2)$ operations.

- One can use (+) to improve the operation count.

- Define the node polynomial
$$\omega(x) = \prod_{i=0}^{m} (x - x_i). \qquad \left(\omega_{m+1}(x) \right)$$

Then
$$l_j(x) = \prod_{\substack{i=0\\ j\neq i}}^{n} \frac{x-x_i}{x_j-x_i} = \omega(x) \frac{\lambda_j}{x-x_j}, \qquad \lambda_j = \frac{1}{\prod\limits_{\substack{i=0\\ i\neq j}}^{m} (x_j - x_i)}$$

Since $l_j(x) = \dfrac{\omega(x)}{\omega'(x_j)(x-x_j)}$, we have $\lambda_j = \dfrac{1}{\omega'(x_j)}$.

$$\text{cond}(x,m,f) \equiv \frac{\sum_{j=0}^{m} |l_j(x) f_j|}{|L_m(x)|}$$

5m op

$$(++) \quad \boxed{L_m(x) = \omega(x) \sum_{j=0}^{m} \frac{\lambda_j}{x-x_j} \cdot f_j} \qquad (f_j = f(x_j))$$

modified Lagrange formula, Jacobi 1825

- If weights λ_j are known $\Rightarrow$ only $O(n)$ operations to evaluate $L_m(x)$
- Comp. weights requires $O(m^2)$ op. $\to$ needs to be done just once
- For special grids $\to$ λ_j are known.

- forward stable $\dfrac{|L_m(x) - \widehat{L_m}(x)|}{|L_m(x)|} \leq O(m)\, u\, \text{cond}(x,m,f)$

HIGHAM 2004

It holds that
$$1 = \sum_{j=0}^{m} l_j(x) = \omega(x) \sum_{j=0}^{m} \frac{\lambda_j}{x-x_j} \Rightarrow \omega(x) = \frac{1}{\sum_{j=0}^{m} \frac{\lambda_j}{x-x_j}}$$

Use (++),
$$\boxed{L_m(x) = \frac{\sum_{j=0}^{m} \frac{\lambda_j}{x-x_j} \cdot f_j}{\sum_{j=0}^{m} \frac{\lambda_j}{x-x_j}}}$$

barycentric formula interpolation

$\to$ used by chebfun for interpolation in Chebyshev points

$\dfrac{|L_m(x) - \widehat{L_m}(x)|}{|L_m(x)|} \leq O(m)\, u\, [\text{cond}(x,m,f) + \Lambda_m]$

$\to$ numerically stable (forward) for reasonable sets of int. points

- for a general distribution of points better to use (++)

Note

- Newton interpolation formula – another form $\to$ we can add interpolation points, using divided differences. It can be shown

$$L_{m+1}(x) = L_m(x) + \prod_{j=0}^{m} (x-x_j)\, f[x_0, x_1, \dots, x_m, x_{m+1}]$$

$\hookrightarrow$ square brackets

divided diff. of order m+2

where $f[x_j] \equiv f(x_j)$,
$$f[\underbrace{x_j, x_{j+1}, \dots, x_{j+k+1}}_{k+1}] = \frac{f[\overbrace{x_{j+1}, \dots, x_{j+k+1}}^{k}] - f[\underbrace{x_j, \dots, x_{j+k}}_{k}]}{x_{j+k+1} - x_j}$$

Then, by induction,
$$\boxed{L_m(x) = f(x_0) + (x-x_0) f[x_0,x_1] + (x-x_0)(x-x_1) f[x_0,x_1,x_2] + \dots + \prod_{j=0}^{m-1} (x-x_j) f[x_0,\dots,x_m].}$$

the Newton interpolation formula

[Powell, Theorem 5.2]

The error in polynomial interpolation

We are interested in the error function,

$$e(x) = f(x) - L_m(x), \quad a \le x \le b.$$

Let $x_0, \dots, x_m$ and $x \ne x_i$, $i = 0, \dots, m$ are given. Define $x_{m+1} = x$ and use the relation

$$f(x) = L_{m+1}(x) = L_m(x) + \underbrace{\prod_{j=0}^{m} (x - x_j)}_{\omega_{m+1}(x)} f[x_0, x_1, \dots, x_m, x]$$

Hence

$$\boxed{e(x) = f(x) - L_m(x) = \omega_{m+1}(x) f[x_0, x_1, \dots, x].}$$

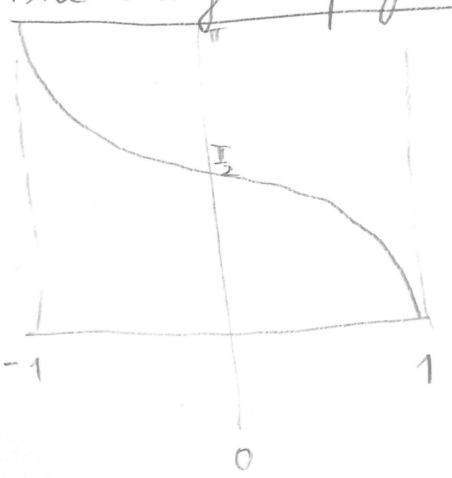
Let $f \in C^{(m+1)}[a,b]$. Then $\forall x \in [a,b]\ \exists \xi_x \in (a,b)$:

$$\boxed{e(x) = \frac{f^{(m+1)}(\xi_x)}{(m+1)!} \omega_{m+1}(x).}$$

Proof. Use Rolle's theorem several times. ∎

- By the choice of x_i we influence the size of $|e(x)|$.
- What is a reasonable distribution of interpolation points that gives "small" $\|e(x)\|_\infty$?
- It is hard to predict the influence of x_i to $f^{(m+1)}(\xi_x)$ → suppose that it is bounded.
- Choose x_i such that $\|\omega(x)\|_\infty = \|\prod_{j=0}^{m}(x - x_j)\|_\infty$ is minimal.
  ↓
  Chebyshev points.

The Chebyshev polynomials



$$[-1, 1] \underset{\longleftarrow}{\xrightarrow{\cos^{-1}}} [0, \pi]$$

$$x \longleftrightarrow \theta$$

$$x = \cos\theta$$

↳ nonlinear, but one-to-one (bijection)

Consider, on $[0,\pi]$, the functions

$$1, \cos\theta, \cos 2\theta, \dots, \cos k\theta,$$

and transform them onto $[-1,1]$ using the $\cos^{-1}$ mapping. On $[-1,1]$, we get

$$\underbrace{1}_{T_0(x)=1}, \underbrace{\cos(\cos^{-1}(x))}_{T_1(x)=x}, \underbrace{\cos(2\cos^{-1}(x))}_{T_2(x)}, \dots\dots, \underbrace{\cos(k\cos^{-1}(x))}_{T_k(x)}$$

- It holds that

$$\underbrace{\cos((k+1)\theta)}_{T_{k+1}(x)} + \underbrace{\cos((k-1)\theta)}_{T_{k-1}(x)} = \underbrace{2\cos(\theta)\cos(k\theta)}_{2\,x\,T_k(x)}$$

Therefore

$T_0(x)=1$

$T_1(x)=x$

and $\boxed{T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)}$, $k=1,\dots$

$\Rightarrow T_k(x)$ are polynomials called Chebyshev polynomials.

Properties

- Roots

$$0 = T_k(x) = \cos(\underbrace{k\cos^{-1}(x)}_{\text{should be odd multiples of } \frac{\pi}{2}})$$

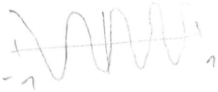
$0,\pi \to$ odd multiples of $\frac{\pi}{2}$ on $[0,k\pi]$

$T_k(x_j)=0 \iff k_j\cos^{-1}(x_j) = \frac{2j-1}{2}\pi$, $j=1,\dots,k$

$$\boxed{x_j = \cos\left(\frac{2j-1}{2k}\pi\right)}, \quad j=1,\dots,k$$

k real distinct roots in $(-1,1)$

- Extrema occurs for



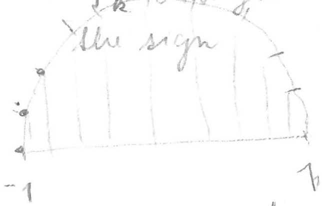
$$\boxed{x_j = \cos\left(\frac{j\pi}{k}\right)}, \quad j=0,\dots,k$$

$\rightarrow T_k$ changes the sign

Obvious $\max\limits_{x\in[-1,1]} T_k(x) = 1$, $\min\limits_{x\in[-1,1]} T_k(x) = -1$

$1 = T_k(x) \iff k\cos^{-1}(x) = 0, 2\pi, 4\pi, \dots$

$-1 = T_k(x) \iff k\cos^{-1}(x) = \pi, 3\pi, 5\pi, \dots$



equidistantly spaced points on the unit circle

- Orthogonality of polynomials with respect to the weight

$$w(x) = \frac{1}{\sqrt{1-x^2}}$$

$$\int_{-1}^{1} T_m(x)\,T_k(x)\,\frac{1}{\sqrt{1-x^2}}\,dx = -\int_{\pi}^{0}\cos(m\theta)\cos(k\theta)\,d\theta = \begin{cases} 0 & k \neq m \\ \pi & k=m=0 \\ \frac{\pi}{2} & k=m\neq 1 \end{cases}$$

substitution

$x = \cos\theta$

$\theta = \cos^{-1}x$

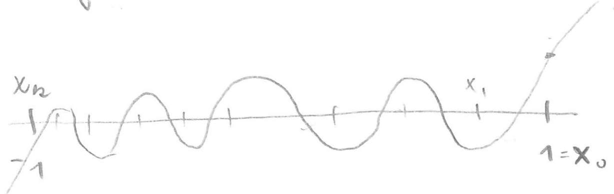
$d\theta = -\frac{dx}{\sqrt{1-x^2}}$

Theorem The monic polynomial $W_k(x) \equiv \frac{1}{2^{k-1}} T_k(x)$ minimizes the maximum on $[-1,1]$ over all monic polynomial of degree k.

Proof. By contradiction. Let there exist a monic pol. $V_k(x)$ s.t.

$$(*) \qquad \max_{x\in[-1,1]} |V_k(x)| < \max_{x\in[-1,1]} |W_k(x)|.$$

- The maximum of $|W_k(x)|$ on $[-1,1]$ occurs at the points $x_j = \cos\left(\frac{j\pi}{k}\right)$. $j=0,\dots,k$.



- at x_0, using $(*)$, $W_k(x_0) > V_k(x_0)$
- at x_1 $W_k(x_1) < V_k(x_1)$ etc

$\Rightarrow$ $W_k(x) - V_k(x)$ changes the sign k times $\Rightarrow$ k roots

but $\deg(W_k - V_k) < k$ $\to$ contradiction. ▨

Back to interpolation

W_k solves $\to \min_{p\in\Pi_k} \underbrace{\max_{x\in[-1,1]} |p(x)|}_{\|p\|_\infty}$

- If we choose interpolation points as

$$(**) \qquad x_j = \cos\left(\frac{2(m-j)+1}{2(m+1)}\pi\right), \quad j=0,\dots,m,$$

$\hookrightarrow$ ordered roots of T_{m+1}

then $\boxed{\omega_{m+1}(x) = \frac{1}{2^m} T_{m+1}(x)}$

has the smallest possible $\|\cdot\|_\infty$ (the smallest deviation from 0).

Points $(**)$ are called <u>Chebyshev points of the first kind</u>.

- In practical computations we use Chebyshev extreme points

$$(***) \qquad x_j = \cos\left(\frac{m-j}{m}\pi\right), \quad j=0,\dots,m$$

$\to$ analogous approximation properties, contain boundary points

<u>Chebyshev points of second kind</u> $\to$ we just say <u>Chebyshev points</u>

- For $[a,b]$ we use the linear transformation onto $[-1,1]$

$$y = \frac{x(b-a)+a+b}{2}$$

$$\underset{x}{[-1,1]} \longleftrightarrow \underset{y}{[a,b]}$$

Then,

$$y_j = \lambda + \mu\cos\left(\frac{m-j}{m}\pi\right),$$

$$\lambda = \tfrac{1}{2}(a+b)$$
$$\mu = \tfrac{1}{2}(b-a).$$

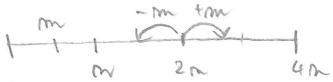
Chebyshev interpolants and the barycentric interpolation formula

Cheb. interpolants .... Lagrange interpolant in Cheb. points.

Aliasing of Cheb. polynomials

<u>Lemma</u>. Let $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j=0,\dots,n$. Then, for $0 \le m \le n$, it holds that

$$T_m(x_j) = T_{2n-m}(x_j) = T_{2n+m}(x_j).$$



$4n, 6n \dots$

<u>Proof</u>. $T_m(x) = \cos(m \cos^{-1}(x))$.

- $T_m(x_j) = \cos\left(m \frac{j\pi}{n}\right)$
- $T_{2n-m}(x_j) = \cos\left((2n-m)\frac{j\pi}{n}\right) = \cos\left(2j\pi - m\frac{j\pi}{n}\right) = \cos\left(m\frac{j\pi}{n}\right)$
  $= \cos\left(m \frac{j\pi}{n}\right)$. ▨
- $T_{2n+m}(x_j) = \dots$

In particular, for $m = n-1$ we obtain

$$T_{n-1}(x_j) = T_{n+1}(x_j), \quad j=0,\dots,n \qquad T_{n-1} \quad T_{n} \quad T_{n+1}$$

$$\boxed{\underbrace{\frac{1}{2^n}\left(T_{n+1}(x) - T_{n-1}(x)\right)}_{\substack{\text{is monic} \\ =0 \text{ at } x_j,\ x_j = 0,\dots,n}} = \omega_{n+1}(x)}$$

Recall

$$\ell_j(x) = \omega_{n+1}(x)\frac{\lambda_j}{x-x_j}, \qquad \lambda_j = \frac{1}{\omega'_{n+1}(x_j)}.$$

Therefore

$$\lambda_j = \frac{2^n}{T'_{n+1}(x_j) - T'_{n-1}(x_j)} \quad \text{if } x_j \text{ are Cheb. points.}$$

<u>Lemma</u> Let $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j=0,\dots,n$. Then

$$\lambda_j = \frac{2^{n-1}}{n}(-1)^j \quad \text{for } j=1,\dots,n-1, \text{ and}$$

$$\lambda_j = \frac{1}{2}\left[\frac{2^{n-1}}{n}(-1)^j\right] \quad \text{for } j=0 \text{ and } j=n.$$

<u>Proof</u>. If $x \in (0,1)$, then

- $\frac{d}{dx} T_k(x) = \frac{d}{dx}\left(\cos(k\cos^{-1}(x))\right) = \boxed{k\,\frac{\sin(k\cos^{-1}(x))}{\sin(\cos^{-1}(x))}}$.

[use $(\cos^{-1}(x))' = -\frac{1}{\sqrt{1-x^2}}$ and $\in (0,\pi)$

$\sqrt{1-x^2} = \sqrt{\sin^2\theta} = |\sin\theta| = |\sin(\cos^{-1}(x))| = \sin(\cos^{-1}(x))$]

- Then

$$T'_{n+1}(x_j) - T'_{n-1}(x_j) = (n+1)\frac{\sin\left((n+1)\frac{j\pi}{n}\right)}{\sin\left(j\frac{\pi}{n}\right)} - (n-1)\frac{\sin\left((n-1)\frac{j\pi}{n}\right)}{\sin\left(\frac{j\pi}{n}\right)}$$

$$= \frac{(n+1)\sin\left(j\pi + \frac{j\pi}{n}\right) - (n-1)\sin\left(j\pi - \frac{j\pi}{n}\right)}{\sin\left(\frac{j\pi}{n}\right)}$$

j even

$$= \frac{(n+1)\sin\left(\frac{j\pi}{n}\right) - (n-1)\sin\left(-\frac{j\pi}{n}\right)}{\sin\left(\frac{j\pi}{n}\right)} = 2n$$

j odd

$$= \frac{(n+1)\sin\left(\frac{-j\pi}{n}\right) - (n-1)\sin\left(\frac{j\pi}{n}\right)}{\sin\left(\frac{j\pi}{n}\right)} = -2n$$

- T_k' is continuous, differentiable

for $x = \pm 1$ take the limit

$$\underset{\substack{\shortparallel \\ x_0}}{T_k'(1)} = \lim_{x\to 1} k \frac{\sin(k\cos^{-1}(x))}{\sin(\cos^{-1}(x))} = \lim_{\theta\to 0+} k\frac{\sin(k\theta)}{\sin\theta} = k^2$$

$$\underset{\substack{\shortparallel \\ x_n}}{T_k'(-1)} = \lim_{\theta\to\pi-} k\frac{\sin\theta k}{\sin\theta} = \begin{cases} k^2 & \text{if } k \text{ is even} \\ -k^2 & \text{if } k \text{ is odd}\end{cases}$$

and

$$T'_{n+1}(\overset{1}{\overset{\shortparallel}{x_0}}) - T'_{n-1}(x_0) = (n+1)^2 - (n-1)^2 = 4n = 2(-1)^j 2n \quad \text{for } j=0,$$

$$T'_{n+1}(\underset{\substack{\shortparallel \\ -1}}{x_n}) - T'_{n-1}(x_n) = \left\langle \begin{matrix} 4n & (\text{even } n) \\ -4n & (\text{odd } n)\end{matrix} \right\rangle = 2(-1)^j 2n \quad \text{for } j=n. \quad \blacksquare$$

<u>Theorem</u> The polynomial interpolant through data $\{f_j\}$ at the Chebyshev points $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j = 0, \dots, n$ is

$$\boxed{L_n(x) = \frac{\sum_{j=0}^{n}{}' \frac{(-1)^j f(x_j)}{x - x_j}}{\sum_{j=0}^{n}{}' \frac{(-1)^j}{x - x_j}}},$$

The primes on the summation signs signify that the terms $j=0$ and $j=n$ are multiplied by $\frac{1}{2}$.

<u>Proof.</u> Barycentric formula

$$L_n(x) = \frac{\sum_{j=0}^{n} \frac{\lambda_j}{x - x_j} f_j}{\sum_{j=0}^{n} \frac{\lambda_j}{x - x_j}} \quad \text{and} \quad \begin{aligned} \lambda_j &= \frac{2^{n-1}}{n}(-1)^j, \quad j = 1, \dots, n-1 \\ \lambda_j &= \frac{1}{2}\frac{2^{n-1}}{n}(-1)^j, \quad j=0, j=n. \end{aligned}$$

$\blacksquare$

- extraordinarily effective $O(n)$ operations
- good num properties

The norm of the Lagrange interpolation operator

For distinct $x_0, \dots, x_n \in [a,b]$ define the Lebesgue function

$$\boxed{\Lambda(x) \equiv \sum_{j=0}^{n} |\ell_j(x)|}$$

$\|\Lambda(x)\|_\infty \overset{\text{def}}{=} \max_{x\in[a,b]} \Lambda(x)$ ... the Lebesgue constant

Theorem Let $x_0, \dots, x_n$ be distinct points from $[a,b]$. Consider the Lagrange interpolation operator $X : C[a,b] \to P_n$,
$X : f \mapsto L_n$, $L_n(x_i) = f(x_i)$, $i = 0, \dots, n$. Then

$$\|X\|_\infty = \|\Lambda\|_\infty .$$

[Cheney, Light 2000, p. 13]

Proof. It holds that

$\leq$

$$\|X(f)\|_\infty = \max_{x\in[a,b]} |(Xf)(x)| = \max_{x\in[a,b]} \left|\sum_{i=0}^{n} f(x_i)\ell_i(x)\right|$$
$$\leq \max_{x\in[a,b]} \sum_{i=0}^{n} |f(x_i)||\ell_i(x)| \leq \|\Lambda\|_\infty \|f\|_\infty$$
$$\Downarrow$$
$$\|X\|_\infty \leq \|\Lambda\|_\infty .$$

$\geq$

$$\exists \xi \in [a,b] : \Lambda(\xi) = \|\Lambda\|_\infty .$$

- Choose $f \in C[a,b]$: $\|f\|_\infty = 1$ and $f(x_i) = \operatorname{sgn} \ell_i(\xi)$.
- Then
$$\|X\|_\infty \geq \|Xf\|_\infty \geq (Xf)(\xi) = \sum_{i=0}^{n} f(x_i)\ell_i(\xi) = \sum_{i=0}^{n} |\ell_i(\xi)| = \|\Lambda\|_\infty .$$
$\square$

Second motivation for the choice of x_i:

$$\|X(f) - f\|_\infty \leq (1 + \|X\|_\infty) \min_{p\in P_n} \|f - p\|_\infty$$

↳ choose x_i such that $\|X\|_\infty$ is small.

$\Lambda_n \geq \dots$

Theorem (Faber 1914). Suppose that $\forall n$ we are given $n+1$ distinct interpolation points from $[a,b]$. Then $\exists f \in C[a,b]$: $\|f - L_n f\|_\infty \to \infty$ as $n \to \infty$.
(no sets of interpolation points can lead to convergence $\forall f \in C[a,b]$) $\square$

a consequence of a more general theorem we discussed

Hence

$$\limsup_{n\to\infty} \Lambda_n = \infty .$$

present first $\Lambda_n \geq \dots$, then Fabers' thm follows.

Theorem. Denote Λ_n the Lebesgue constant for a given set of $n+1$ distinct interpolation points in $[-1,1]$. Then

$$\Lambda_n \geq \frac{2}{\pi}\log(n+1) + 0.52125\ldots$$

[Bernstein 1912, Jackson 1913, Faber 1914]

$\rightarrow \frac{2}{\pi}\left(\gamma + \log\left(\frac{4}{\pi}\right)\right)$ → natural log

$\hookrightarrow$ Euler's constant $0.577\ldots$ $\lim_{n\to\infty}\left(-\log n + \sum_{k=1}^{n}\frac{1}{k}\right) = -\int_0^\infty e^{-x}\log(x)\,dx$

For Chebyshev points of the first and second kind it holds that

$$\Lambda_n \leq \frac{2}{\pi}\log(n+1) + 1.$$

For equispaced points

$$\Lambda_n > \frac{2^{n-2}}{n^2}.$$

↳ to know more about convergence, study

Convergence of Chebyshev interpolants

Chebyshev polynomials form an orthogonal system of functions in $C[-1,1]$,

$$\langle p,q\rangle = \int_{-1}^{1} p\cdot q\cdot\frac{1}{\sqrt{1-x^2}}\,dx$$

One can project any $f\in C[-1,1]$ to this system

the best approx of f in the induced norm

$$f \longmapsto \sum_{i=0}^{\infty}\frac{\langle f,T_i\rangle}{\langle T_i,T_i\rangle}T_i = \frac{2}{\pi}\sum_{i=1}^{\infty}\langle f,T_i\rangle T_i + \frac{1}{\pi}\langle f,T_0\rangle.$$

[TREF. Thm 3.1]

Theorem If f is Lipschitz continuous on $[-1,1]$, it has a unique representation as a Chebyshev series,

$$f(x) = \sum_{i=0}^{\infty} a_i T_i(x), \quad a_i = \frac{2}{\pi}\langle f,T_i\rangle, \quad a_0 = \frac{1}{\pi}\langle f,T_0\rangle$$

which is absolutely and uniformly convergent.

Proof. Based on Dini-Lipschitz criterion ... later. for periodic functions

Continue 11-12

[TREF. p.25]

Def $f_n(x) = \sum_{i=0}^{n} a_i T_i(x)$, a_i defined as above
truncation Chebyshev projection

(*12) Λ_n

Page about connections 11

~~[see Lebe]~~

~~**Theorem** If f is Lipschitz continuous on $[-1,1]$, then~~

$$\cancel{\|f-p_n\|_\infty \leq 2\|f-f_n\|_\infty,}$$

~~where p_n is the Chebyshev interpolant.~~

~~Consequence: p_n is as good as f_n. (for f Lipschitz)~~

~~Cheb series converges~~ ~~constant 2~~ $\Rightarrow$ ~~p_n converges absolutely and uniformly to f~~

$$\cancel{\|f-p_n\|_\infty \to 0}$$

Done! p_n is as good as f_n in the sense of bounds!

27.10.2016

Theorem [Speed of convergence for differentiable function]

Let $f \in C[a,b]$. Let $f^{(k)}$, $k = 1, \ldots, \nu - 1$, $\nu \in \mathbb{N}$ ($\nu \geq 1$) be absolutely continuous and suppose that $f^{(\nu)}$ is of bounded variation V.
Then $\forall n > \nu$ its Chebyshev interpolants satisfy

$$\boxed{\|f - p_n\|_\infty \leq \frac{4V}{\pi \nu} \cdot \frac{1}{(n-\nu)^\nu}.}$$

Recall:

- Continuity $\forall \varepsilon \, \forall x \, \exists \delta : |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$
- absolute continuity $\forall \varepsilon \, \exists \delta :$ $\forall$ finite sequence of disjoint subintervals (x_k, y_k)

$$\sum_k |y_k - x_k| < \delta \Rightarrow \sum_k |f(y_k) - f(x_k)| < \varepsilon.$$

- total variation

$$V_a^b(f) = \sup_P \sum |f(x_{i+1}) - f(x_i)|$$

P: partition of $[a,b]$, $P = \{x_0, \ldots, x_{n_P}\}$

$$V_a^b(f) = \int_a^b |f'| \, dx \quad \text{if } f \text{ is differentiable}$$

$\downarrow$ a measure of the length of a curve
(usually, we measure the length by $\int_a^b \sqrt{1 + |f'|^2} \, dx$)

Continuous diff

$\Downarrow$ Lipschitz continuous $\overset{\text{easy}}{\Rightarrow}$ absolutely continuous $\Rightarrow$ bounded variation

Chebfun

- divide interval into subintervals
- on each subinterval $\to$ determine n such that $\dfrac{\|f - f_n\|_\infty}{\|f\|_\infty} \simeq \varepsilon_m$
- use Chebyshev interpolant of degree n p_n

($n \log(n)$ op. FFT)

$\to$ $\boxed{\text{work with } p_n \text{ instead of } f}$

variable chebfun $\to$ store $\begin{cases} \cdot \text{ subintervals} \\ \cdot \text{ degrees of } p_n \\ \cdot \text{ coefficients of Cheb. projection} \end{cases}$

$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$, $p_n(x) = \sum_{k=0}^{n} c_k T_k(x)$ $\hookrightarrow = ?$

$f(x) = \sum_{k=0}^{\infty} a_k T_k(x)$

[Tref. 4.2]

Theorem. Let f be Lipschitz continuous on $[-1,1]$, $n \geq 1$. Then

$2n \pm k$

$$(*) \quad \begin{cases} c_0 = a_0 + a_{2n} + a_{4n} + \dots \\ c_k = a_k + (a_{k+2n} + a_{k+4n} + \dots) \\ \qquad\quad + (a_{-k+2n} + a_{-k+4n} + \dots) \\ c_n = a_n + a_{3n} + a_{5n} + \dots \end{cases} \qquad k = 1, \dots, n-1$$

$m=0$ $\quad$ $m=1$

$\Rightarrow$ $\boxed{\text{unification are all coef of } f.}$

Proof.

- Realise, f has a unique Cheb. series, converges absolutely
  $\Rightarrow$ we can rearrange the terms without affecting conv
- $c_0, \dots, c_n$ are well-defined, the corresponding series in $(*)$ converge, e.g.
  $$c_0 = \sum_{i=0}^{\infty} a_{2ni} = \sum_{i=0}^{\infty} a_{2ni} T_{2ni}(1),$$ similarly the other coef
- at Chebyshev point $x_j = \cos\left(\frac{j\pi}{n}\right)$, $j = 0, \dots, n-1$ $\longrightarrow$ aliasing, $0 \leq m \leq n$
  $$T_m(x_j) = T_{2n-m}(x_j) = T_{2n+m}(x_j) = \dots$$

so that
$$\underbrace{\sum_{k=0}^{n} c_k T_k(x_j)}_{\text{a pol. of degree at most } n} = [a_0 + a_{2n} + \dots] T_0(x_j) + \sum_{k=1}^{n-1} [a_k + \dots] T_k(x_j) + [a_n + a_{3n} + \dots] T_n(x_j)$$

aliasing $\rightarrow$ can change the index to the index of a coef in brackets

$$= \sum_{k=0}^{\infty} a_k T_k(x_j) = f(x_j).$$

$\Rightarrow \sum_{k=0}^{n} c_k T_k(x_j)$ is interpolant in Cheb points, unique $= p_n$ ∎

Note

$$p_n(x) = \sum_{k=0}^{n} c_k T_k(x) = \sum_{j=0}^{n} a_j T_j(x) + \underbrace{\sum_{m=1}^{\infty} \left(a_{2mn} T_0 + \sum_{k=1}^{n-1} (a_{2mn+k} + a_{2mn-k}) T_k + a_{2mn-n} T_n \right)}_{\sum_{j=n+1}^{\infty} a_j T_{m(n,j)} \text{ správný index}}$$

$0 \leq m \leq n$

(*) po sloupcích
přenásob každé
T_j a sečti po sloupcích

$m = (j+n-1) \pmod{2n} - (n-1)$

hence

$$f - f_m = \sum_{j=m+1}^{\infty} a_j T_j(x)$$

$$f - p_m = \sum_{j=m+1}^{\infty} a_j \left(T_j(x) - T_{m(m,j)}(x) \right)$$

f Lipschitz

$$\left.\begin{aligned} \|f - f_m\|_\infty &\le \sum |a_j| < \infty \\ \|f - p_m\|_\infty &\le 2 \sum_{m+1} |a_j| < \infty \end{aligned}\right\}$$ absolutely convergent !

The Lebesgue constant of the Cheb. projection

$$\underset{C[a,b]}{R_m : f} \longmapsto \underbrace{R_m f}_{f_m} = \underset{\text{Cheb. projection}}{\sum_{i=0}^{n} a_i T_i}$$ → linear operator, projection ($T_0, \dots, T_n$ form a basis of polynomials) ↓ orthogonal projection

It can be shown that $$\|R_m\|_\infty = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin((m+\frac{1}{2})\theta)}{\sin(\frac{\theta}{2})} \right| d\theta$$

technical and $\|R_m\|_\infty \le \frac{4}{\pi^2} \log(m+1) + 3$, $\|R_m\|_\infty \ge \frac{4}{\pi^2} \log(m+1)$

best polynomial approx

In summary, we know that

$$\|f - f_m\|_\infty \le \left(4 + \frac{4}{\pi^2} \log(m+1)\right) \|f - p_m^*\|_\infty$$

$$\|f - p_m\|_\infty \le \left(2 + \frac{2}{\pi} \log(m+1)\right) \|f - p_m^*\|_\infty .$$

* Note → Linear projections 14

Convergence recall LC ⇒ AC ⇒ BV

Theorem $\nu \ge 0$ … integer. Let f and its derivatives through $f^{(\nu-1)}$ be absolutely continuous on $[-1,1]$ ans suppose the νth derivative $f^{(\nu)}$ if of bounded variation V. Then for $k > n$, the Chebyshev coef. of f satisfy $V_{-1}^{1}(f) \le V$.

$$|a_k| \le \frac{2V}{\pi k(k-1)\cdots(k-\nu)} \le \frac{2V}{\pi(k-\nu)^{\nu+1}} .$$

If f is analytic with $|f(x)| \le M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then $\forall k \ge 1$

open set

$$|a_0| \le M, \qquad |a_k| \le \frac{2M}{\rho^k} .$$ → [Bernstein 1912]

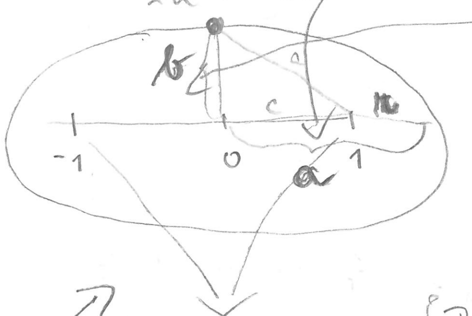
Proof Book Trefethen Thm 7.1 using complex analysis Thm 8.1 | Paper Tref. 2008 → standard real anal.

→ use integral representation of a_k and integration by part

Note. Bernstein ellipses ($c+a=a$)

$2c+2b$ = const
$2a$ = const

dolžina glavne polosi
a ... semi-major axes
vedejša
b ... semi-minor
c ... linear eccentricity   ekscentričnost

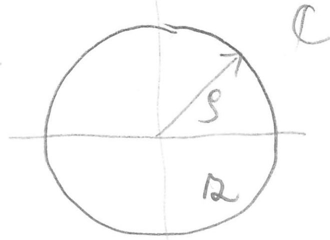
E_ρ = 

$c = 1$

$a^2 - b^2 = 1$

Define $\rho = a + b$

[Bernstein 1912]



foci

images of the circle of radius ρ in $\mathbb{C}$ under the Joukowsky map $\quad x = \frac{1}{2}(z + z^{-1})$.

Theorem With the notation and asumptions of the previous theorem, it holds that for any $n > \nu$

$$\| f - f_n \|_\infty \le \frac{2V}{\pi\nu (n-\nu)^\nu}, \qquad \sim \frac{1}{n^\nu} \text{ provelhon}$$

$$\| f - p_n \|_\infty \le \frac{4V}{\pi\nu (n-\nu)^\nu}.$$

If f is analytic with $|f(x)| \le M$ in the open region bounded by E_ρ, then

$$\| f - f_n \|_\infty \le \frac{2M\rho^{-n}}{\rho - 1}, \quad \| f - p_n \|_\infty \le \frac{4M\rho^{-n}}{(\rho - 1)}.$$

Proof : Follows from

$$\| f - f_n \|_\infty \le \sum_{k=n+1}^{\infty} |a_k|, \quad \| f - p_n \|_\infty \le 2 \sum_{k=n+1}^{\infty} |a_k|,$$

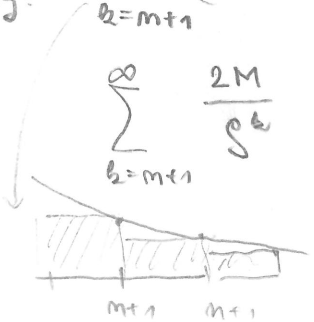
and bounding and summing $|a_k|$ by previous Theorems. ▨

Note : If $f \in C[-1,1]$ is analytic, it can be analytically continued to a neighborhood of $[-1,1]$ in the complex plane. The bigger the neighborhood, the faster the convergence.

E.g.

$$\sum_{k=n+1}^{\infty} \frac{1}{(k-\nu)^{\nu+1}} \le \int_n^\infty \frac{1}{(x-\nu)^{\nu+1}}\,dx = \frac{1}{\nu(n-\nu)^\nu}$$

$$\sum_{k=n+1}^{\infty} \frac{2M}{\rho^k} = 2M \rho^{-(n+1)} \sum_{j=0}^{\infty} \rho^{-j} = 2M \frac{\rho^{-(n+1)}}{1-\rho^{-1}} = 2M \frac{\rho^{-n}}{\rho - 1}$$



area

(To see connections)

Note It can be shown the following theorem [Powell 17.4, p. 208]

Theorem If L is any bounded linear operator from $C[-1,1]$ to P_n that satisfies the projection condition $Lp = p \; \forall p \in P_n$, then the inequality

$$\|L\|_\infty \geq \frac{1}{2} \|R_0 + R_n\|_\infty \text{ holds.} \qquad \|R_0\|_\infty = 1$$

$$\Rightarrow \|L\|_\infty \geq \frac{2}{\pi^2} \ln(n+1) - \frac{1}{2}$$

Notes on computation

- evaluate f at Cheb. point $\cos\left(\frac{k\pi}{n}\right)$
- evaluate Cheb. interp in $O(n)$ op.
- need to compute a_k ... coef of Cheb projection → control accuracy; → manipulate with f work

$$\frac{2}{\pi} \int_{-1}^{1} f \, T_k(x) \frac{dx}{\sqrt{1-x^2}} = \frac{2}{\pi} \int_0^{\pi} f(\cos\theta) \cos(k\theta) \, d\theta = \frac{1}{\pi} \int_{-\pi}^{\pi} \bigcirc \, d\theta = \frac{1}{\pi} \int_0^{2\pi} \bigcirc \, d\theta$$

$x = \cos\theta$
$d\theta = -\frac{dx}{\sqrt{1-x^2}}$

cos is symmetric

2π periodic

standard form.

$\boxed{\text{approx. int. numerically}}$

N... powers of 2

$$\frac{1}{\pi} \int_0^{2\pi} g(\theta) \, d\theta \approx \frac{1}{\pi} \sum_{k=0}^{N-1} g\left(\frac{2k\pi}{N}\right) \frac{2\pi}{N} = \frac{2}{N} \sum_{k=0}^{N-1} g\left(\frac{2k\pi}{N}\right)$$

0 ——— 2π

$\frac{2\pi}{N} = h$

$$\begin{bmatrix} f(\cos(\frac{2 \cdot 0 \cdot \pi}{N})) \\ \vdots \\ f(\cos(\frac{2 \cdot (N-1) \pi}{N})) \end{bmatrix} \qquad \begin{bmatrix} \cos(k \frac{2 \cdot 0 \pi}{N}) \\ \vdots \\ \cos(k \cdot \frac{2 \cdot N \cdot \pi}{N}) \end{bmatrix}$$

$$a_k \approx \frac{2}{N} \sum_{j=0}^{N-1} f\left(\cos\left(\frac{2j\pi}{N}\right)\right) \cdot \cos\left(k \frac{2j\pi}{N}\right) = \frac{2}{N} \langle \vec{f}, \vec{C}_k \rangle$$

→ Discrete cosine transf

can be computed using FFT.

$$\begin{bmatrix} a_0 \\ \vdots \\ a_k \end{bmatrix} \approx \frac{2}{N} \begin{bmatrix} - \vec{C}_0 - \\ \vdots \\ - \vec{C}_k - \end{bmatrix} \begin{bmatrix} \vec{f} \end{bmatrix}$$

$\cos\left(\frac{2\pi jk}{N}\right)$ → $N \log N$ op.

choose $k = N-1$

$N = 2^m$

- We can evaluate f_n using CLENSHAW'S Algorithm.

$\sum_{k=0}^{n} c_k F_k(x)$ ↳ 3 term recurrence $O(n)$ op.

- solving nonlinear equations

$f \Leftrightarrow f_n$, solve $f_n(x) = 0$

↳ polynomial represented in basis T_i

$$f_n(x) = \sum_{k=0}^{n} a_k T_k(x)$$

↳ switch to each nodes

$f_n(x)$ is characteristic polynomial of and minimal

$$C = \begin{bmatrix} 0 & 1 & & \\ 1/2 & 0 & 1/2 & \\ & \ddots & \ddots & 1/2 \\ & & 1/2 & 0 \end{bmatrix} - \frac{1}{2a_n} \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ a_0 & a_1 \ldots & a_{n-1} \end{bmatrix}$$

→ $\boxed{\text{Colleague matrix}}$

→ just eigenvalue computation

Recall if $q(x) = \sum_{k=0}^{n} q_k x^k$ → is char & min pol. of

$$\begin{bmatrix} 0 & & & -a_0 \\ 1 & \ddots & & \vdots \\ & \ddots & 0 & \\ & & 1 & -a_{n-1} \end{bmatrix} = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \cdots & 0 & a_0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1} \end{bmatrix}$$

→ $\boxed{\text{Companion matrix}}$

- integrating

$$\int_{-1}^{1} f(x)\,dx \approx \int_{-1}^{1} p_n(x)\,dx$$

→ to be interpolatory quadrature rule

→ Clenshaw - Curtis quadrature

if $p_n = \sum_{k=0}^{n} c_k T_k$, then

$$\int_{-1}^{1} p_n(x)\,dx = \sum_{k=0}^{n} c_k \underbrace{\int_{-1}^{1} T_k(x)\,dx}_{0 \text{ for } k \text{ odd}, \ \frac{2}{1-k^2} \text{ k even}} = 2 \sum_{\substack{k=0 \\ k \text{ even}}}^{n} \frac{c_k}{1-k^2}$$

we know, how to get c_k from a_k.

v GQ uvažujeme uzly své dány

1. Koeficienty
2. Vyhodnocení Σ . Clenshaw
3. Kořeny
4. Integrace

Proof Consider

$$v(x) = \begin{bmatrix} T_0(x) \\ \vdots \\ T_{n-1}(x) \end{bmatrix} \qquad T_0(x) = 1 \quad T_1(x) = x$$
$$T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)$$

$$C\,v(x) = \begin{bmatrix} T_1(x) \\ \frac{1}{2}T_0(x) + \frac{1}{2}T_2(x) \\ \vdots \\ \frac{1}{2}T_{n-3}(x) + \frac{1}{2}T_{n-1}(x) \\ \frac{1}{2}T_{n-2} - \frac{1}{2a_n}\sum_{k=0}^{n-1} a_k T_k \end{bmatrix} = \begin{bmatrix} x T_0 \\ \vdots \\ x T_{n-2} \\ x T_{n-1}(x) - \frac{1}{2a_n}\underbrace{\sum_{k=0}^{n} a_k T_k}_{f_n(x)} \end{bmatrix}$$

$\pm \frac{1}{2} T_n(x)$

$$= x\,v(x) - \frac{1}{2a_n}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ f_n(x) \end{bmatrix}$$

- x is root of $f_n(x)$ => x is an eigenvalue of C
- if f_n has n distinct roots => they are precisely the eigs. of C
- if f_n has multiple roots -> consider small perturbations of coef so that the roots become distinct

$$\tilde{f}_n \leftrightarrow \tilde{C}$$

roots of polynomials as well as eigenvals of C are continuous functions of coef -> multiplicities must be preserved in the limit as perturbations -> 0.

Remark about numerical stability of computation of roots

$$q(x) = \psi_n(x) + \sum_{k=0}^{n-1} a_k \psi_k(x) \qquad \psi_i \ldots \text{ basis}$$

↓ from coef

C
colleague, companion, comrade

↓ QR alg, backward stable

$\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n$

$\tilde{\lambda}_i$ are eigs. of $C+E$

$$\frac{\|E\|}{\|C\|} = O(\varepsilon)$$

Are $\tilde{\lambda}_i$ roots of a nearby polynomial $\tilde{q}$

Denote $\|q\| = \sqrt{\sum_{k=0}^{n-1} a_k^2}$

It holds that

$$\frac{\|\tilde{q} - q\|}{\|q\|} = O(\varepsilon)\|q\|$$

if this is big, the computation of roots need not be backward stable.

holds for monomial & Cheb. basis

Problem formulation

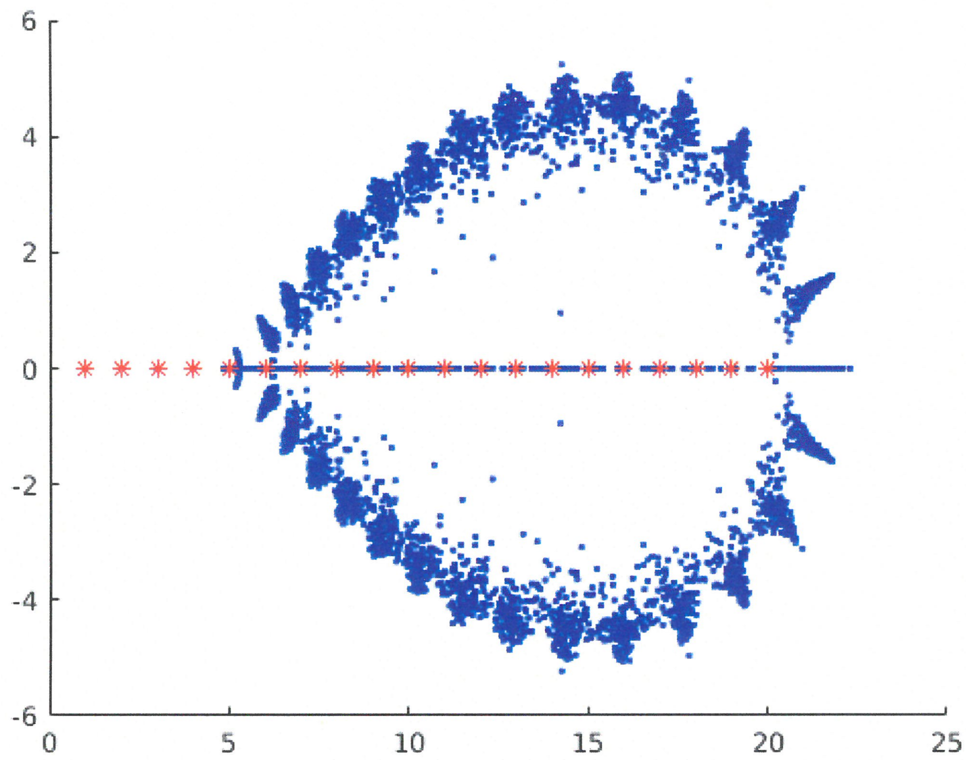
- $p(x) = \sum a_i x^i$
- $q(x) = \sum b_i T_i$
- generate randomly a_i, b_i
- Where are the roots placed?

The Perfidious Polynomial

Wilkinson's polynomial, paper 1970

$$p(x) = a_0 + a_1 x + \cdots + a_{19} x^{19} + x^{20} = \prod_{k=1}^{20} (x - k)$$

$$\tilde{p}(x) \equiv \tilde{a}_0 + \tilde{a}_1 x + \cdots + \tilde{a}_{19} x^{19} + x^{20}$$



$$\tilde{a}_k = a_k(1 + 10^{-10} \cdot \mathrm{randn}(-1, 1))$$