Approximation problems numapprox.blogspot.com

approximate f by a member of $\mathcal{A}$

How to choose an approximation?

What is f? A function (given explicitly or implicitly – a solution of a problem), vector etc.
some data

What is $\mathcal{A}$? A simple, parameter dependent function
Polynomials, rational functions, piecewise pol. fcions ...

How to choose an approximation? Interpolation, the smallest distance measured by a metric or a norm <u>BEST APPROXIMATION</u>

1. EXISTENCE OF BEST APPROXIMATIONS

Approximation in a metric space

$\mathcal{B}$ ... a metric space, $d(x,y)$ .... the distance function, $\mathcal{A} \in \mathcal{B}$

Def. We define $a^* \in \mathcal{A}$ to be a best approximation of $f \in \mathcal{B}$, if

$$(*) \qquad d(a^*, f) \leq d(a, f) \quad \forall a \in \mathcal{A}.$$

(?) When there exists a best approximation?

proof without using result on continuity. Otherwise follows from continuity & compactness.

<u>Theorem</u> If $\mathcal{A} \in \mathcal{B}$ is a compact set, then, for $\forall f \in \mathcal{B}$, there exist an element $a^* \in \mathcal{A}$, such that the condition $(*)$ holds.

<u>Proof.</u> Recall: A $\boxed{\text{compact}}$ set $\mathcal{A}$ in a metric space is bounded and closed, every sequence of points in $\mathcal{A}$ has a subsequence which converges to a point of $\mathcal{A}$.

- Let $d^* = \inf_{a \in \mathcal{A}} d(a, f)$
- From the definition of an infimum, $\exists$ a sequence $\{a_i\} \in \mathcal{A}$ which gives
$$\lim_{i \to \infty} d(a_i, f) = d^*$$
- Compactness .. $\{a_i\}$ has a subsequence $\{\tilde{a}_j\}$, $\tilde{a}_j \longrightarrow \tilde{a} \in \mathcal{A}$
- $d(\tilde{a}, f) \leq \underbrace{d(\tilde{a}, \tilde{a}_j)}_{\to 0} + \underbrace{d(\tilde{a}_j, f)}_{\to d^*}$ $\Rightarrow$ It cannot hold that $d(\tilde{a}, f) > d^*$ & from def d^* $d^* \leq d(\tilde{a}, f)$ $\Rightarrow$ $d^* = d(\tilde{a}, f)$ ☒

→ left hand side is arbitrarily close to d^*

EXAMPLE: Open unit ball is not compact
best app. may not exist

Properties of metric spaces → not sufficiently strong for most of our work

From now on $\mathcal{B}$ ... a normed linear space, $\|\cdot\|$ ... a norm

→ we can measure distance using $d(x,y) = \|x-y\|$

Theorem: If $\mathcal{A}$ is a finite dimensional linear space in a normed lin. space $\mathcal{B}$, then, $\forall f \in \mathcal{B}$ there exists an element of $\mathcal{A}$ that is a best app from $\mathcal{A}$ to f.

Proof.

- Let $\mathcal{A}_0 \equiv \{ a \in \mathcal{A} : \|a\| \leq 2\|f\| \}$
- $\mathcal{A}_0$ is compact (bounded, closed, $\dim \mathcal{A} < \infty$)
- It is not empty ($0 \in \mathcal{A}$)
- By previous Theorem, $\exists a_0^* \in \mathcal{A} : \|a-f\| \geq \|a_0^* - f\| \quad \forall a \in \mathcal{A}_0$

  since $0 \in \mathcal{A}_0 \Rightarrow \|f\| \geq \|a_0^* - f\|$

- If $a \in \mathcal{A}$ & $a \notin \mathcal{A}_0$, then

  $\|a-f\| \geq \|a\| - \|f\| > 2\|f\| - \|f\| = \|f\| \geq \|a_0^* - f\|$.

- We have shown

  $\|a-f\| \geq \|a_0^* - f\| \quad \forall a \in \mathcal{A} \Rightarrow a_0^*$ is a best app from $\mathcal{A}$ to f ▨

THE L_p-norms

We usually consider $\mathcal{A} \subset C([a,b])$, $f \in C([a,b])$ - continuous problems
or $\mathcal{A} \subset \mathbb{R}^m$, $f \in \mathbb{R}^m$ - discrete problems

Norms that are used most frequently: L_p norms for $p = 1, 2, \infty$

- $\|f\|_p \equiv \left[\int_a^b |f(x)|^p \, dx \right]^{1/p} \quad 1 \leq p \leq \infty, \quad \|f\|_\infty \equiv \max_{a \leq x \leq b} |f(x)|$
- $f = [f_1, \dots, f_m]^T, \quad \|f\|_p \equiv \left[\sum_{i=1}^m |f_i|^p \right]^{1/p}, \quad \|f\|_\infty \equiv \max_{1 \leq i \leq m} |f_i|$

1 .... suitable for fitting to discrete data
2 ... occurs naturally in theoretical studies of Hilbert spaces
when $\mathcal{A}$ is a linear space → best appr. → system of lin. eq.
∞ ... suitable for uniform approximation (uniform or minimax norm)

Theorem: $\forall e \in C([a,b])$ the inequalities

$$\|e_1\| \leq (b-a)^{1/2} \|e_2\| \leq (b-a) \|e\|_\infty \quad \text{hold.}$$

Without a proof (use Cauchy-Schwarz). Discrete case → we have also opposite inequalities → with N.

Continuous problems $\to$ small in $\|\cdot\|_1$ or $\|\cdot\|_2$ $\not\Rightarrow$ small in $\|\cdot\|_\infty$

EXAMPLE

$$f(x) \equiv 1 \quad \text{on } [0,1] \quad \xrightarrow[l_2]{\text{approximate}} \quad x^\lambda, \quad \lambda > 0$$

error function $\quad e_\lambda(x) = 1 - x^\lambda$

$$\|e_\lambda\|_1 = \frac{\lambda}{\lambda+1}, \quad \|e_\lambda\|_2 = \left[\frac{2\lambda^2}{(\lambda+1)(2\lambda+1)}\right]^{1/2}, \quad \|e_\lambda\|_\infty = 1$$

for $\lambda \to 0$. $\quad \|e_\lambda\|_1 \to 0$ $\quad$ but $\|e_\lambda\|_\infty = 1$ $\quad \boxtimes$

$\|e_\lambda\|_2 \to 0$

A geometrical view of best approximation

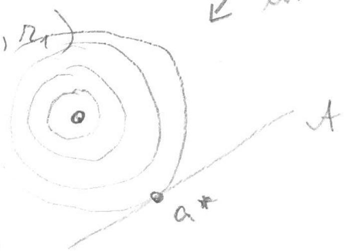
Consider a balls with radius r which center is at f

$$N(f,r) \equiv \{ g \in B : \|g-f\| \le r \}.$$

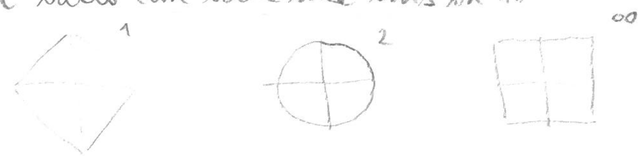
Balls of different radii

$$r_1 > r_0 \Rightarrow N(f,r_0) \subset N(f,r_1)$$

is allowed to grow until it touches A



But unit balls can look like this in $\mathbb{R}^2$



Uniqueness of a best app?

THE UNIQUENESS OF BEST APPROXIMATIONS

Some conditions for uniqueness depend on the convexity of the distance function and the convexity of A.

Recall $\quad$ convexity of a set

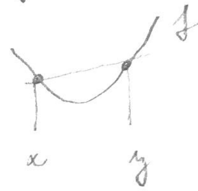
$\alpha x + (1-\alpha) y \in \Omega$

$\alpha \in (0,1)$

strictly convex

if $\alpha x + (1-\alpha) y$ are interior points of Ω. I.e., the boundary does not contain a segment of a straight line.

function convex



$f(\alpha x + (1-\alpha) y) \le \alpha f(x) + (1-\alpha) f(y)$

$\downarrow$

$<$ $\quad$ strictly convex

Theorem $\quad$ Let B be a normed linear space. Then $\forall f \in B$ and any $r > 0$, the ball $\quad N(f,r) = \{ x : \|x-f\| \le r,\ x \in B \}$ is convex.

Proof. $\quad$ We ask $\quad x \in N(f,r),\ y \in N(f,r) \Rightarrow \alpha x + (1-\alpha) y \in N(f,r)$?
$\alpha \in (0,1)$

$$\|\alpha x + (1-\alpha) y - f\| = \|\alpha(x-f) + (1-\alpha)(y-f)\|$$
$$\le \alpha \|x-f\| + (1-\alpha)\|y-f\| \le r \quad \boxtimes$$

Consequence. If $A \subset B$ is convex and if $\exists$ a best app. to f, then the set of best approximations is <u>convex</u>.

<u>Def.</u> The norm is defined to be <u>strictly convex</u> iff $N(0,1)$ is strictly convex.

<u>Geometrical view</u>

If the boundary of either A or $N(f, r^*)$ is curved, and if both sets are convex $\Rightarrow$ there is only one point of contact.

$\downarrow$ Will be formulated precisely in the following two theorems

<u>Theorem</u> Let A be a compact and strictly convex set in a normed linear space B. Then, $\forall f \in B$, there is just one best approximation from A to f.

<u>Proof.</u>

- A is compact $\Rightarrow$ existence
  Denote $h^* = \min\limits_{a \in A} \| a - f \|$ $\to$ consider only the nontrivial case $f \notin A$, i.e., $h^* > 0$.

- with contradiction. Let s_0, s_1 be different best apps.

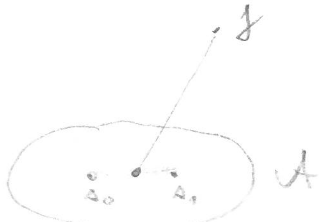
$$\| \underbrace{\tfrac{1}{2}(s_0+s_1)}_{} - f \| \leq \tfrac{1}{2}\|s_0 - f\| + \tfrac{1}{2}\|s_1 - f\| = h^*$$

$\Downarrow$ it is also a best app and $=$ (or use the previous consequence - the set of best app is convex)

- let $\lambda \in [0,1]$ is the largest number such that

$$s = \underbrace{\tfrac{1}{2}(s_0+s_1)}_{\text{interior point}} + \lambda \underbrace{\left[f - \tfrac{1}{2}(s_0+s_1) \right]}_{\text{a direction}}$$

is in A. A is strictly convex



Since A is compact, λ is well-defined.
$\lambda > 0$ ($\frac{1}{2}(s_0+s_1)$ is an interior point)

- It holds that

$$\| s - f \| = \| (1-\lambda)(\tfrac{1}{2}(s_0+s_1) - f) \| = (1-\lambda) h^*$$

and $\lambda > 0$, $h^* > 0$, $s \in A$ $\Rightarrow$ $\|s - f\| < h^*$ $\Rightarrow$ contradiction $\blacksquare$

<u>Theorem</u> Let A be a convex set in a normed linear space B whose norm is strictly convex. Then, $\forall f \in B$, there is at most one best approximation from A to f.

<u>Proof.</u> • Let s_0 and s_1 be different best app. $h^* = \min\limits_{a \in A} \| a - f \|$.

- $\| s_0 - f \| = \| s_1 - f \| = h^* \Rightarrow s_0 \in N(f, h^*)$
  $s_1 \in N(f, h^*)$

$\Rightarrow N(f, h^*)$ is strictly convex $\Rightarrow$ $\frac{1}{2}(s_0 + s_1)$ is an interior point

and $\| \frac{1}{2}(s_0+s_1) - f \| < h^*$.

- But $\frac{1}{2}(s_0+s_1) \in \mathcal{A}$ $\Rightarrow$ contradiction ▨
  (from the convexity)

Minkowski's theorem

We usually use $\mathcal{A}$ ... finite dimensional linear subspace

$\Rightarrow$ we have convexity

considering L^p spaces

It is important to know whether the norm of $\mathcal{B}$ is strictly convex.

Which norms are strictly convex?

in general L_p-norms are strictly convex for $1 < p < \infty$. Use Minkowski inequality. Rudin

Theorem The 2-norm of the linear space $\mathcal{B}$ is strictly convex when $\mathcal{B}$ is either $C([a,b])$ or $\mathbb{R}^n$.

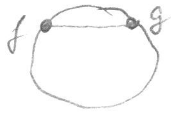
- Every Hilbert space is strictly convex
  see, e.g., the book by Rivlin, p. 6.

Proof. We use the connection with a scalar product

$$(f,g) = \int_a^b fg \, dx \qquad (y,z) = \sum_{i=1}^n z_i y_i$$

$$\|f\|_2^2 = (f,f) \qquad \|y\|_2^2 = (y,y)$$

- Let f and g be any two distinct points of $\mathcal{B}$, $\|f\|_2 = \|g\|_2 = 1$.
  It is sufficient to prove



$$\|\alpha f + (1-\alpha) g\|_2 < 1 \quad \forall \alpha \in (0,1)$$

- $$\|\alpha f + (1-\alpha) g\|_2^2 + \underbrace{\alpha(1-\alpha)\|f-g\|_2^2}_{\text{we add something positive}} =$$

$$= \alpha^2 + 2\alpha(1-\alpha)(f,g) + (1-\alpha)^2 + \alpha(1-\alpha)(1 - 2(f,g) + 1) = 1. \ ▨$$

The 1- and ∞- norms are not strictly convex.

Example: $\mathcal{B} = C[-1,1]$, $f \equiv 1$, $\mathcal{A} = \{\lambda(1+x) : \lambda \in \mathbb{R}\}$

$$\min_{\lambda \in \mathbb{R}} \|1 - \lambda(1+x)\|_\infty = 1$$

$\hookrightarrow$ this norm $= 1 \quad \forall \lambda \in [0,1]$ ▨

$\Rightarrow$ we have non-uniqueness

- In many important cases, e.g. $\mathcal{B} = C[a,b]$, $\mathcal{A} = \mathcal{P}_n$ alg. pols. of degree $\leq n$

Consider 1- or ∞-norm $\rightarrow$ one can prove uniqueness of the best approximation, although these norms are not strictly convex $\rightarrow$ "Uniqueness" also depends on properties of $\mathcal{A}$ and f. ▨