Finite Element Method 1 - NMNV405
Homework on 7 October 2020

1. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz-continuous boundary and let $\Gamma \subset \partial\Omega$ be a subset of the boundary of Ω having a positive surface measure. Consider any $p \in [1, \infty)$ and prove that there is a constant C such that

$$\|u\|_{0,p,\Omega} \leq C \left(|u|_{1,p,\Omega} + \left| \int_\Gamma u \, \mathrm{d}\sigma \right| \right) \qquad \forall\, u \in W^{1,p}(\Omega).$$

2. Consider the boundary value problem

$$- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) + c\, u = f \qquad \text{in } \Omega\,,$$

$$\sum_{i,j=1}^{n} n_i\, a_{ij} \frac{\partial u}{\partial x_j} + h\, u = g \qquad \text{on } \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz-continuous boundary, $a_{ij} \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, $h \in L^\infty(\partial\Omega)$, and $g \in L^2(\partial\Omega)$. We assume that the matrix $(a_{ij})_{i,j=1}^n$ is uniformly positive definite a.e. in Ω, $c \geq 0$ a.e. in Ω, and $h \geq h_0 = \text{const.} > 0$ on $\partial\Omega$. Derive the variational formulation of the above boundary value problem. The difference to what was discussed during the tutorial is that the boundary condition contributes to the bilinear form by the term $\int_{\partial\Omega} h\, u\, v \, \mathrm{d}\sigma$. The test function space V is $H^1(\Omega)$. Verify that the variational formulation satisfies all the assumptions of the abstract variational problem.

3. Consider the boundary value problem

$$- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial u}{\partial x_j} \right) = f \qquad \text{in } \Omega\,,$$

$$\sum_{i,j=1}^{n} n_i\, a_{ij} \frac{\partial u}{\partial x_j} = g \qquad \text{on } \partial\Omega\,,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz-continuous boundary, $a_{ij} \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, and $g \in L^2(\partial\Omega)$. We assume that the matrix $(a_{ij})_{i,j=1}^n$ is uniformly positive definite a.e. in Ω and that $\int_\Omega f \, \mathrm{d}x + \int_{\partial\Omega} g \, \mathrm{d}\sigma = 0$ (the latter condition is necessary for the solvability of the considered problem). Derive the variational formulation of the above boundary value problem. Since the solution of the classical formulation is never unique (it is determined up to an arbitrary additive constant), the space for a uniquely solvable weak formulation cannot be $H^1(\Omega)$. One possibility is to use $V = \{v \in H^1(\Omega)\,;\ \int_\Omega v \, \mathrm{d}x = 0\}$. Another possibility is to use the factor space $H^1(\Omega)/P_0(\Omega)$ consisting of classes of functions which differ by a constant. The usual norm of a factor space is $\|\tilde{v}\|_{H^1(\Omega)/P_0(\Omega)} = \inf_{v \in \tilde{v}} \|v\|_{1,\Omega}$ for any $\tilde{v} \in H^1(\Omega)/P_0(\Omega)$. For both choices of V verify that the variational formulation satisfies all the assumptions of the abstract variational problem.