BANKING

T01 – Financial Mathematics

Magda Pečená, Matěj Kuc
Institute of Economic Studies, Faculty of Social Sciences,
Charles University in Prague, Czech Republic

30 September 2020

Annuity

- An annuity is an instrument that generates the same sum of money for a certain period, and the number of instalments is known.
- Examples of an annuity are a mortgage loans or money paid each month to a retiree.

$$PV(I_0) = C \times \left[\frac{1}{r} - \frac{1}{r \times (1+r)^T} \right]$$

PV – PV of instalments
C – instalment
r – interest rate
T – maturity

…if instalments are paid at the end of the period

(if it is paid twice a year: r/2 instead of r and 2t instead of t)

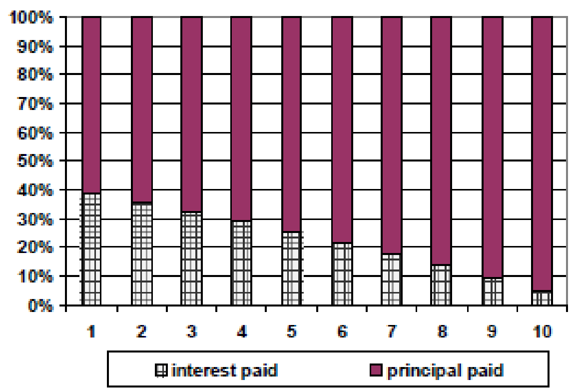
Task 1 (Annuity)

Calculate an instalment for a CZK 1 million loan with 10-year maturity, interest rate at 5% and yearly instalments.

Principal	1 000 000
Interest rate	5%
Maturity	10
Instalments per year	1
Annuity	?

$$C = \frac{PV(I_0)}{\left[\frac{1}{r} - \frac{1}{r \times (1+r)^T}\right]} = \frac{1,000,000}{\left[\frac{1}{5\%} - \frac{1}{5\% \times (1+5\%)^{10}}\right]} = 129,504.57$$

Year	Instalment	Interest paid	Principal paid	Principal left	Discount factor	PV of instalments
1	129,505	50,000	79,505	920,495	0.9524	123,338
2	129,505	46,025	83,480	837,016	0.9070	117,464
3	129,505	41,851	87,654	749,362	0.8638	111,871
4	129,505	37,468	92,036	657,325	0.8227	106,544
5	129,505	32,866	96,638	560,687	0.7835	101,470
6	129,505	28,034	101,470	459,217	0.7462	96,638
7	129,505	22,961	106,544	352,673	0.7107	92,036
8	129,505	17,634	111,871	240,802	0.6768	87,654
9	129,505	12,040	117,464	123,338	0.6446	83,480
10	129,505	6,167	123,338	0	0.6139	79,505
	1,295,046	295,046	1,000,000			1,000,000



Task 2 (mortgage)

You want to buy a flat worth CZK 2,000,000. A bank will provide you with a mortgage of up to 70% of the flat's purchase price. Calculate an instalment for this mortgage provided its 15-year maturity, an interest rate of 6% and monthly instalments.

$$C = \frac{PV(I_0)}{\left[\frac{1}{r/m} - \frac{1}{r/m \times (1 + r/m)^{m \times T}}\right]} = \frac{1400\,000}{\left[\frac{1}{6\%/12} - \frac{1}{6\%/12 \times (1 + 6\%/12)^{12 \times 15}}\right]} = 11{,}814.00$$

Perpetuity

A perpetuity (or perpetual annuity) is an annuity that is payable for a period of time without any fixed end, i.e. its principal is not to be paid. (e.g. a consol bond, common stock)

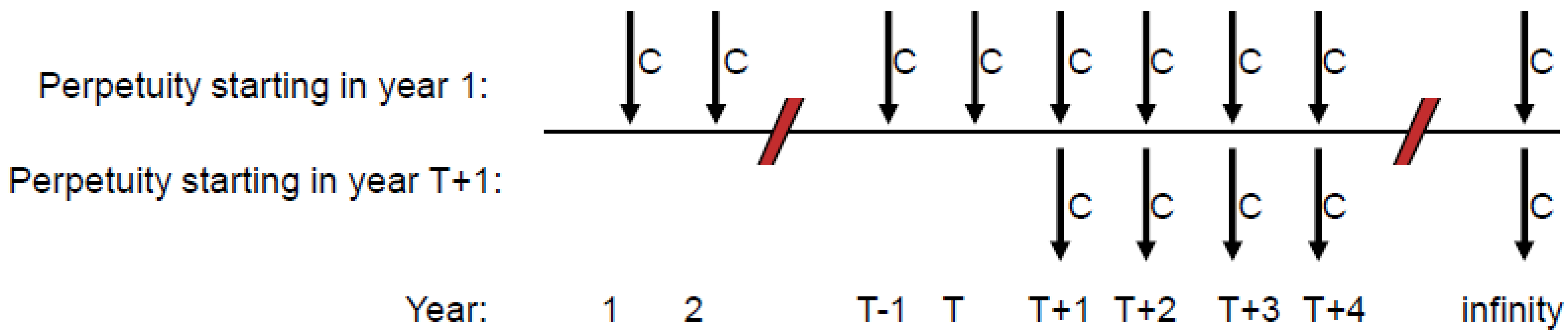
$$PV(I_0) = \frac{C}{r}$$

PV – present value of instalments
C – coupon
r – interest rate

Task 3 (Perpetuity)

Demonstrate that annuity is the difference between two perpetuities in time.

You start getting C in the first year but since year T+1, you have to give the money back.



Hence the present value of a perpetuity starting in year 1 and ending in year T is:

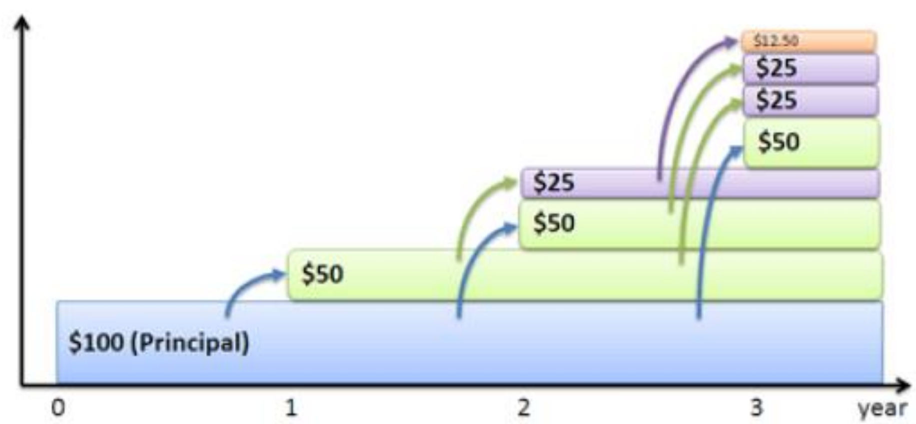
$$PV = \frac{C}{r} - \frac{C}{(1+r)^T r} = C\left(\frac{1}{r} - \frac{1}{(1+r)^T r}\right)$$

Types of interest

1. Simple interest — interest is calculated only on the principal, and no accrued interest occurs.

2. Compound interest (interest upon interest) — interest is payable not only on the principal but also on sums of interest as they accumulate.

Figure: Compound interest



Frequency of interest

In the above-mentioned examples we used interest paid annually. However, in reality other frequencies of interest also occur (e.g. daily, monthly, semiannual etc.)

$$FV(I_0) = I_0 \times (1 + r_{p.a.})^T$$

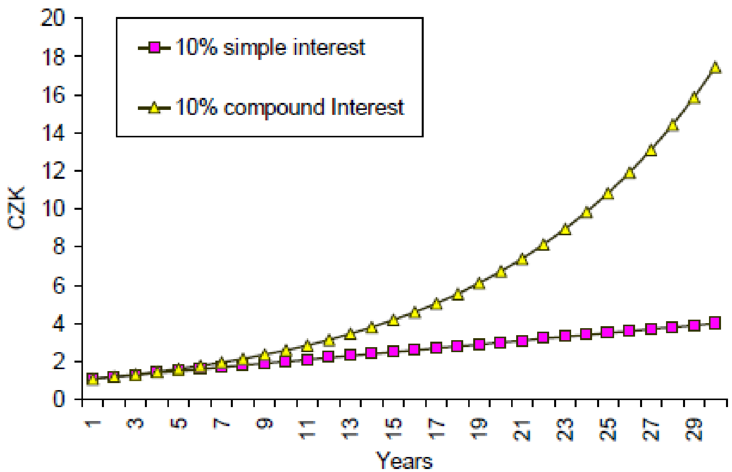
FV	– future value of investment I_0
$r_{p.a.}$	– annual interest rate
T	– maturity

$$FV(I_0) = I_0 \times (1 + \frac{r_{p.a.}}{m})^{T \times m}$$

FV	– future value of investment I_0
r	– interest rate p.a.
T	– maturity
m	– frequency of interest per year

Task 4 (Types of interest)

Show the difference between compound and simple interest based on a deposit yielding 10% and a maturity of 1, 5, 10, 30, 50 and 100 years.



	Simple interest			*Compound interest*			
Year	**Initial Amount**	**Interest**	**Final Amount**	**Initial Amount**	**Interest**	**Final Amount**	**Difference**
1	1	0.10	1.10	1	0.10	1.10	**0.00**
5	1.40	0.10	1.50	1.46	0.15	1.61	**0.11**
10	1.90	0.10	2.00	2.36	0.24	2.59	**0.59**
30	3.90	0.10	4.00	15.86	1.59	17.45	**13.45**
50	5.90	0.10	6.00	106.72	10.67	117.39	**111.39**
100	10.90	0.10	11.00	12,527.83	1,252.78	13,780.61	**13,769.61**

Task 5 (Frequency of interests)

Show that the following formula holds for continuous interest:

$$FV(I_0) = PV(I_0) \times e^{r \times t}$$

1) $$FV = PV \times \left(1+\frac{r}{m}\right)^{m \times T} = PV \times \left[\left(1+\frac{r}{m}\right)^{\frac{m}{r}}\right]^{rT} = PV \times \left[\left(1+\frac{1}{n}\right)^{n}\right]^{rT}$$ where $n = m/r$

2) We know that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e$

3) Hence $$FV = PV \times e^{rt}$$

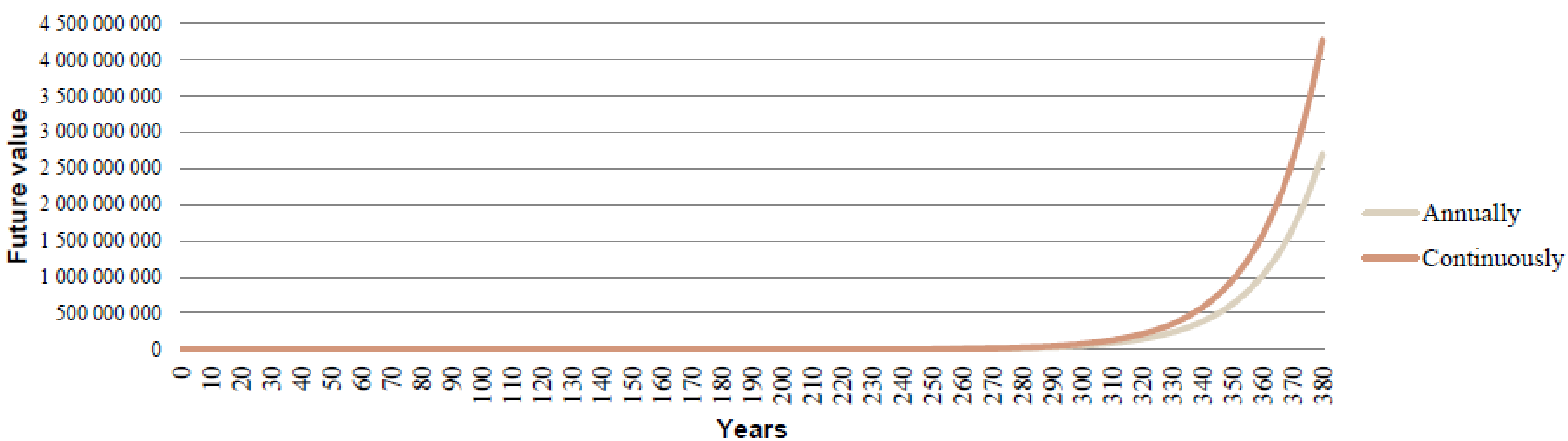
Task 6 (Frequency of interests)

In 1626, Peter Mint, the governor of the colony of New Netherland, bought the island of Manhattan from Indians with beads, cloth and trinkets worth \$24. Find the value of this sum in the year 2006 at 5% compounded

a) continuously and b) annually.

a) $$FV(I_0) = PV(I_0) \times e^{r \times t} = 24 \times e^{5\% \times 380} = \$4,283,575,223$$

b) $$FV(I_0) = I_0 \times (1 + r_{p.a.})^T = 24 \times (1 + 5\%)^{380} = \$2,704,860,603$$



Effective Interest Rate

Considering a variety of interest frequencies, it would be difficult to compare these interest rates. Therefore a new variable has been introduced: effective interest rate, sometimes denoted as annual percentage rate or APR, corresponds to an annual nominal interest rate r_N compounded m-times a year.

$$r_{ef} = \left(1 + \frac{r_N}{m}\right)^m - 1$$

Task 7 (Frequency of interests)

Ota is looking at different banks to find the best investment choice for his deposit. Ota has CZK 100,000 and has a three-year horizon. What bank would you recommend to him? Assume a 15% tax on interest. Here is the table of rates and compounding periods:

Bank	Rate (p.a.)	Compounding	Eff.int.rate
CSOB	7.000%	1	7.000%
Komerční Banka	6.950%	2	7.071%
Česká spořitelna	**6.900%**	**4**	**7.081%**
GE Money	6.850%	12	7.069%

Long-term bonds

A bond is a debt instrument with a maturity of over 1 year. We can distinguish many types of bonds, for example according to:

a) **Coupon:** Zero-coupon vs. coupon bonds

b) **Coupon-rate variability:** Floating-rate vs. fixed-rate bond

c) **Issuer:** Public sector vs. financial institutions vs. companies vs. sovereign

d) **Embedded options:** Callable vs. putable vs. convertible bonds

e) **Maturity:** Short-term vs. medium-term vs. long-term bonds.

Task 8 (Foreign bonds)

A Foreign bonds are the bonds whose issuers are not domiciled in the country where they are issued and traded. Several types of foreign bonds and countries are listed in Table below. Try to assign each bond to each country, where it is traded.

	Bond type		Country of trading
1	Yankee	A	US
2	Samurai	B	Japan
3	Bulldog	C	UK
4	Rembrandt	D	The Netherlands
5	Matador	E	Spain
6	Kangaroo	F	Australia
7	Kauri	G	New Zealand

Table already ordered: 1A, 2B, 3C, 4D, 5E, 6F, 7G

Bond valuation

The valuation of a straight (vanilla) bond that pays constant annual coupons regularly and the principal at the end of maturity.

$$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t} + \frac{M}{(1+r)^T}$$

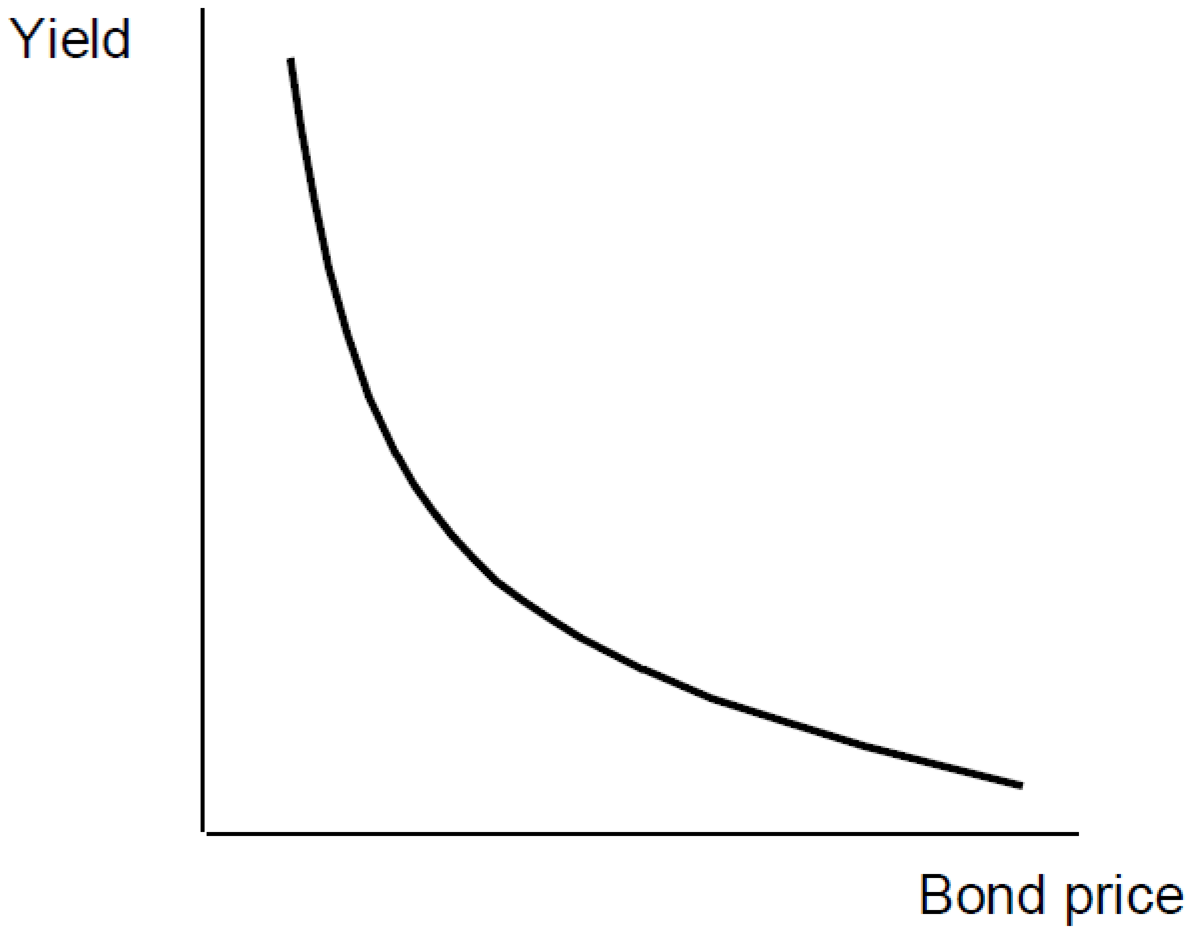
$$P_0 = \sum_{t=1}^{T} \frac{\frac{C_t}{m}}{\left(1+\frac{r}{m}\right)^{m\times t}} + \frac{M}{\left(1+\frac{r}{m}\right)^{m\times T}}$$

P_o – market value of the bond
r – required rate of return
C_t – coupon at time t
M – face value of the bond
T – maturity

$$P_0 = M \times \left[\frac{c}{r} - \frac{c-r}{r\times(1+r)^n}\right]$$

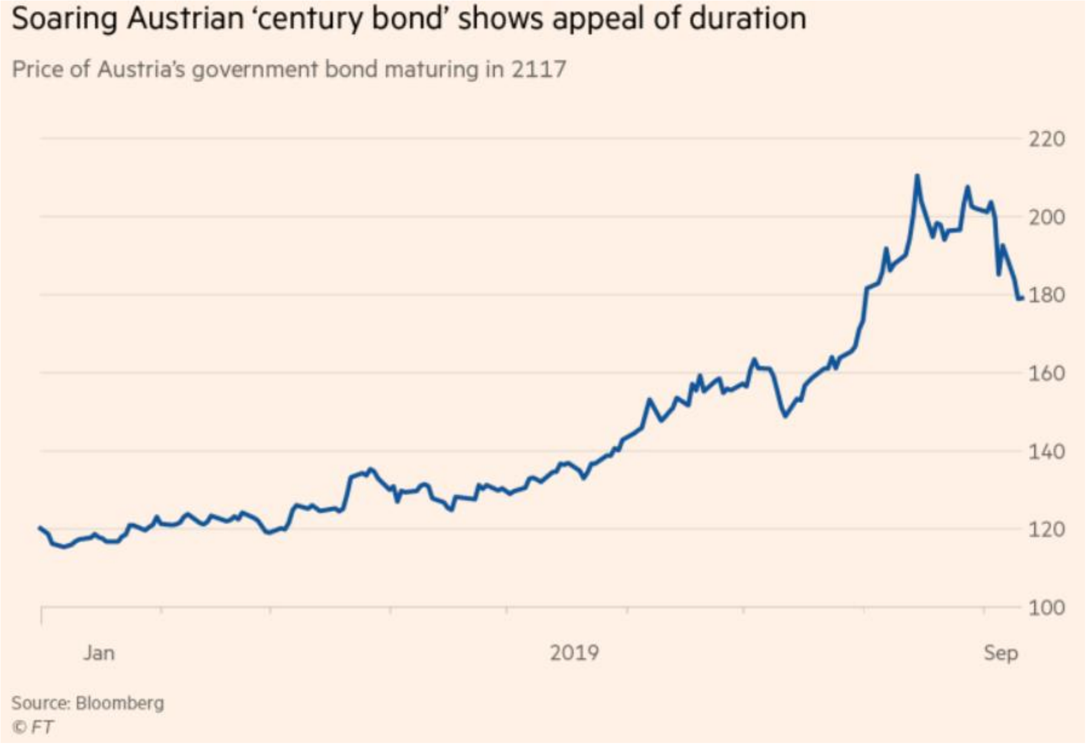
P_o – market value of the bond
M – face value of the bond
r – required rate of return in an interest period
c – coupon rate in an interest period (in %)
T – number of interest periods until the bond's maturity

The inverse (non-linear) relationship between bond prices and yield valuation



The inverse relationship between bond prices and yield valuation – Austrian 100Y GB case

- Austrian government issued 100Y government bond in 2017 (volume EUR 3.5bn; coupon 2.1%)
- Market rates (and expectations!) went down significantly since then…price of a bond increased by 78% in 2019



Vanilla bond yields

Par yield (nominal) yield (c): $$c = \frac{C}{M}$$

Current (flat, running) yield (y): $$y = \frac{C}{P_0}$$

Yield to maturity (YTM): $$P_0 = \sum_{t=1}^{T} \frac{C_t}{(1+YTM)^t} + \frac{M}{(1+YTM)^T}$$

C	– constant coupon
P_o	– market value of the bond
C_t	– coupon at time t
M	– face value of the bond
T	– maturity

Yield to maturity

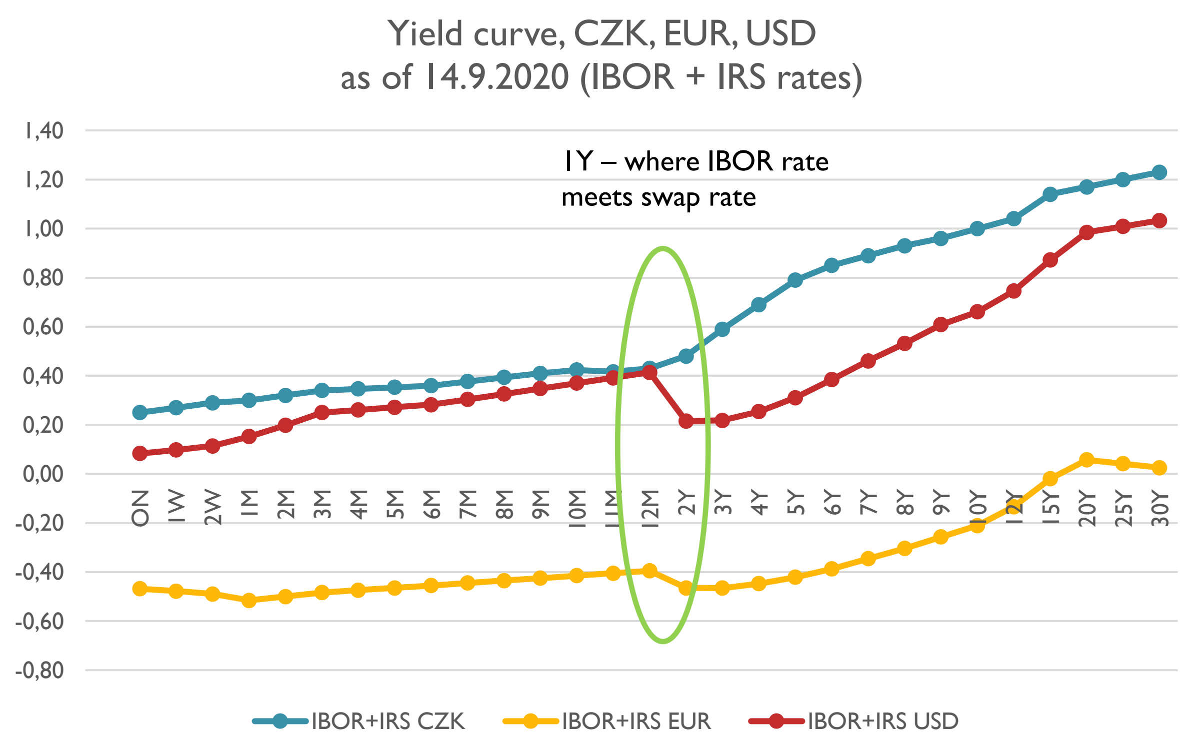
The yield to maturity (YTM) is an average return paid to an investor if he or she holds a bond until its maturity. It is hard to compute without software, so "hand" computing is possible either by iterations (a trial and error method) or approximation (e.g. the Hawawini-Vory's approximate yield to maturity, or AYTM):

$$AYTM = \frac{C + \dfrac{M - P_0}{T}}{0.6 \times P_0 + 0.4 \times M}$$

C – coupon
P_o – purchase price of the bond
M – face value of the bond
T – residual maturity

Yield curve

The yield curve shows the relationship between maturity and yields



Financial Mathematics

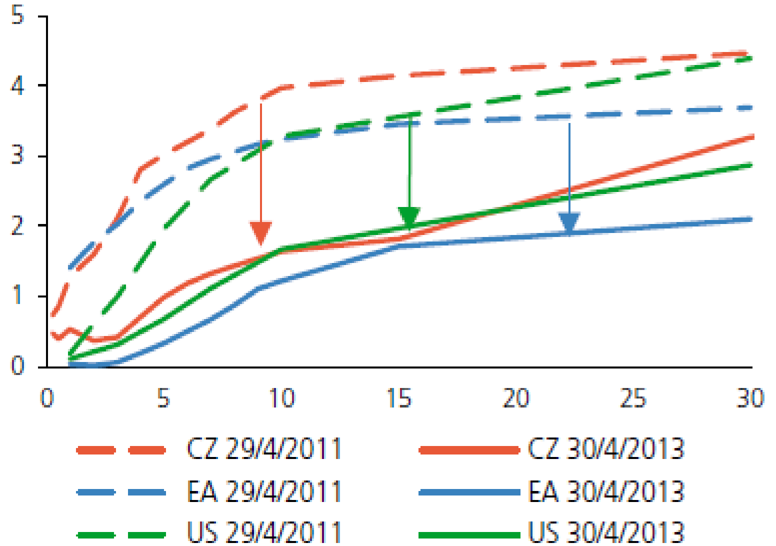
Yield curves in practice (CZGB history)

Yield curves – int. comparison

Decrease in general level of interest rates (2011-2016 -2017)

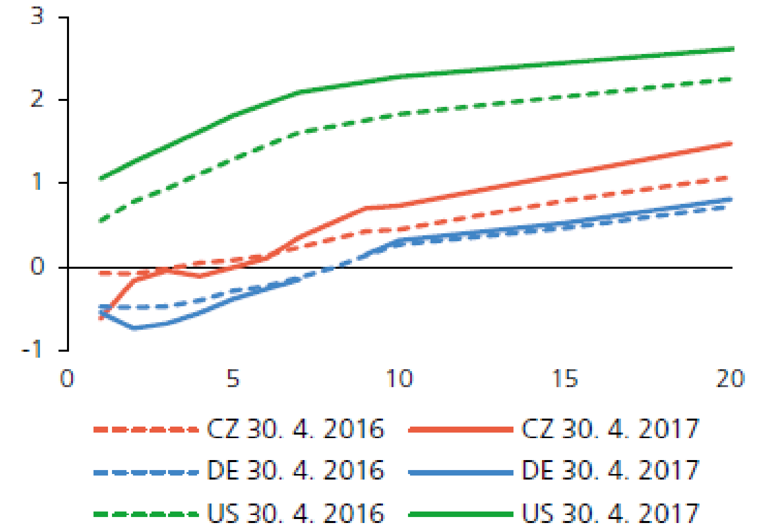
CHART III.6

Movement of government yield curves in selected economies
(x-axis: years; y-axis: %)



Source: Thomson Reuters, Bloomberg LP

Movement of government bond yield curves in selected economies
(x-axis: maturity in years; y-axis: yield in %)



Source: Bloomberg
Note: The yield curves are derived from generic government bond yields.

EA – Euro area

Source: Financial Stability Report 2012/2013, 2016/2017

Accrued interest (1/4)

Accrued interest is a part of the coupon that compensates the Buyer (or the Seller) for non-obtaining of the accrued part of the coupon.

$$P_D = P_C \pm AI$$

P_D – dirty price
P_C – clear (market) price of the bond
AI – accrued interest

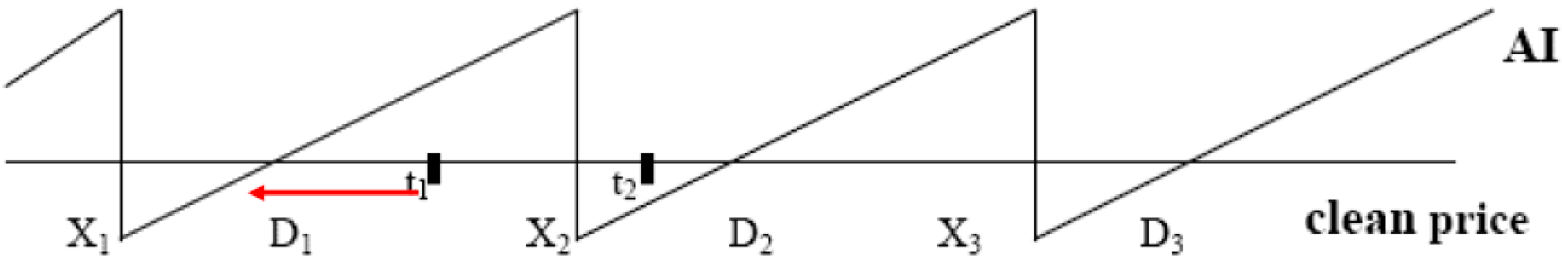
Accrued interest (2/4)

The equation above shows that a bond's dirty price is equal to a bond's price adjusted by AI, which can be both positive and negative based on the date of a bond's sale.

When calculating AI, we should know the ex-dividend day, which is decisive for a coupon payoff for an investor. Whoever owns the bond on that day will receive a coupon.

However, the coupon is to be paid on a dividend day, which usually follows 3–4 days after the ex-dividend day. Two different dates of the sale of the bond are shown in the following figures.

Accrued interest (3/4)

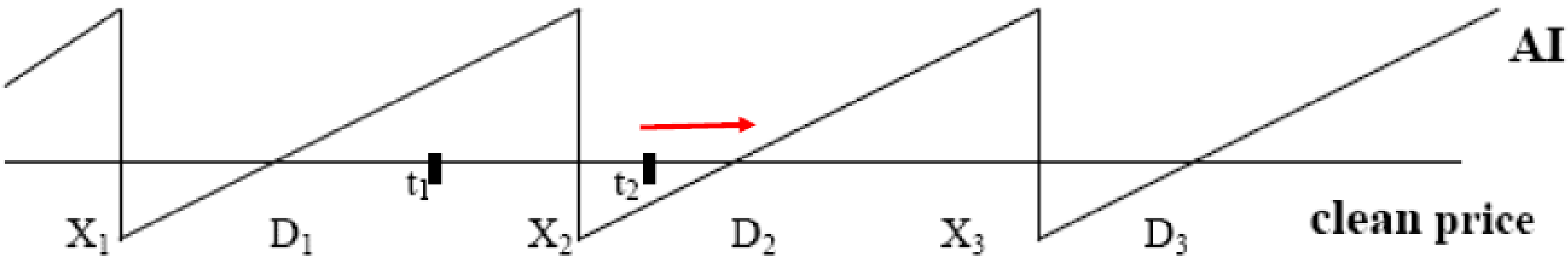


If the deal is done at time t_1, the buyer is to be compensated for holding a bond in period (D_1, t_1)…buyer pays more, i.e. $P_D = P_C + AI$

$$AI = \frac{t_1 - D_1}{360} \times C$$

AI	– accrued interest
t_1	– the date of a bond's sale
D_1	– a dividend day
X_1	– an ex-dividend day
C	– annual coupon

Accrued interest (4/4)



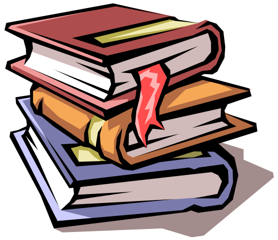
If the deal is done at time t_2, the buyer is to be compensated for holding a bond in period (t_2, D_2)…buyer pays less, i.e. $P_D = P_C - AI$

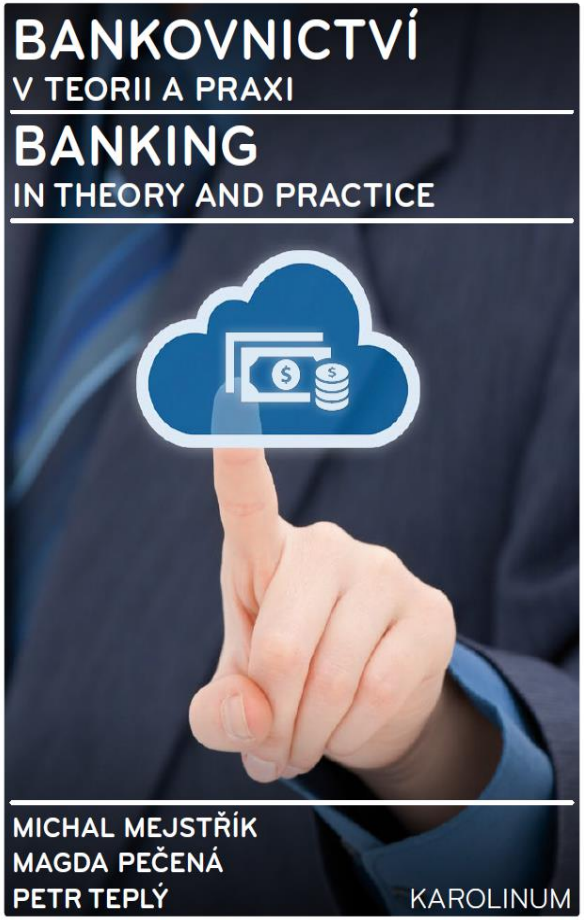
$$AI = \frac{D_2 - t_2}{360} \times C$$

AI	– accrued interest
t_2	– the date of a bond's sale
D_2	– a dividend day
X_2	– an ex-dividend day
C	– annual coupon

Financial Mathematics

Sources





Brealey, R. A., Myers, S. C.: Principles of Corporate Finance, 8th edition, McGraw-Hill, 2006

Capinski, M., Zastawiak M.: Mathematics for Finance, Springer, 2004

Cipra, T.: Matematika cenných papírů, HZ Praha, 2000

Damodaran, A.: Investment Valuation: Tools and Techniques for Determining the Value of Any Asset, John Wiley & Sons, 2002

Dvořák, P., Radová, J., Málek, J.: Finanční matematika pro každého, 6. vydání, Grada, 2008

Jílek, J.: Kapitálový a derivátový trh, Bankovní institut, 1997

Kislingerová, E. et al.: Manažerské finance, 1. vydání, C. H. Beck, 2004

Reilly, F., Brown, K.: Investment Analysis and Portfolio Management, 6th ed, Dryden, New York, 2000

Radová, J. et al.: Finanční matematika pro každého – příklady, 1. vydání, Grada, 2008

Sinkey, J. F.: Commercial Bank Management in the Financial Services Industry, 6th edition, Prentice Hall, 2002

www.csob.cz

www.damodaran.com

Thank you for your attention.