~~Věta T24~~ Věta T 2.4 (aritmetika limit)

Nechť $\lim\limits_{n\to\infty} a_n = A \in \mathbb{R}$ a $\lim\limits_{n\to\infty} b_n = B \in \mathbb{R}$. Potom platí

(i) $\lim\limits_{n\to\infty} a_n + b_n = A + B$

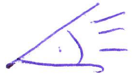
(ii) $\lim\limits_{n\to\infty} a_n \cdot b_n = \lim\limits_{n\to\infty} a_n \cdot \lim\limits_{n\to\infty} b_n$

(iii) pokud $b_n \neq 0$ $\forall n \in \mathbb{N}$ a $B \neq 0$, pak $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$

Pz: MŮŽE BÝT $\exists \lim (a_n + b_n)$, ale $\nexists \lim\limits_{n\to\infty} a_n$, $\lim\limits_{n\to\infty} b_n$

$0 = \lim\limits_{n\to\infty} (-1)^n + (-1)^{n+1}$

TEST $\quad \exists \lim a_n \cdot b_n$, ale $\nexists \lim a_n$ nebo $\lim b_n$

 NPJE

(I) $\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : \ |a_n - A| < \varepsilon$.

(II) $\forall \tilde{\varepsilon} > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : \ |a_n - A| < 2\tilde{\varepsilon}$.

Idea dk: (I) $\Rightarrow$ (II)

(II) $\Rightarrow$ (I) $\quad$ VÍM (II)

Ať $\varepsilon > 0$ volme $\tilde{\varepsilon} = \frac{\varepsilon}{2}$. Pro toto $\tilde{\varepsilon}$ použiji II a dostanu n_0 $\forall n \geq n_0$ $\quad |a_n - A| < 2\tilde{\varepsilon} = 2 \cdot \frac{\varepsilon}{2} = \varepsilon$.

Použiji toto ε pro (I) a hotovo.

Dk: (i) $\lim a_n = A$ $\lim b_n = B \Rightarrow \lim a_n + b_n = A + B$ [5-2]

Nechť $\varepsilon > 0$. Z $\lim\limits_{n\to\infty} a_n = A$ $\exists n_A \in \mathbb{R}$ $\forall n > n_A$: $|a_n - A| < \varepsilon$

Z $\lim\limits_{n\to\infty} b_n = B$ $\exists n_B \in \mathbb{R}$ $\forall n > n_B$: $|b_n - B| < \varepsilon$.

Zvolme $n_0 = \max\{n_A, n_B\}$. Pak $\forall n \geq n_0$ platí

$$|(a_n + b_n) - (A + B)| = |(a_n - A) + (b_n - B)| \leq |a_n - A| + |b_n - B| < \varepsilon + \varepsilon = 2\varepsilon.$$

$\square$

(ii) $\lim a_n = A$ $\lim b_n = B \Rightarrow \lim a_n \cdot b_n = A \cdot B$

$$|a_n \cdot b_n - AB| = |a_n \cdot b_n - b_n \cdot A + b_n \cdot A - AB| \leq$$
$$\leq |a_n \cdot b_n - b_n \cdot A| + |b_n \cdot A - AB| = |a_n - A| \cdot |b_n| + |b_n - b| \cdot |A|$$

omez.

$\exists \lim\limits_{n\to\infty} b_n = B \in \mathbb{R} \overset{V2.2.}{\Longrightarrow} b_n$ je omezená, tedy $\exists K \in \mathbb{R}$ $\forall n \in \mathbb{N}$ $|b_n| \leq K$.

Nechť $\varepsilon > 0$. Z $\lim\limits_{n\to\infty} a_n = A$ $\exists n_A \in \mathbb{R}$ $\forall n \geq n_A$: $|a_n - A| < \varepsilon$

Z $\lim\limits_{n\to\infty} b_n = B$ $\exists n_B \in \mathbb{R}$ $\forall n \geq n_B$: $|b_n - B| < \varepsilon$.

Zvolme $n_0 = \max\{n_A, n_B\}$. Pak $\forall n \geq n_0$ platí

$$|a_n \cdot b_n - AB| = |a_n \cdot b_n - b_n A + b_n A - AB| \leq |a_n \cdot b_n - b_n \cdot A| + |b_n \cdot A - AB|$$
$$\leq |a_n - a| \cdot |b_n| + |b_n - b| \cdot |A|$$
$$\leq \varepsilon \cdot K + \varepsilon |A| = \varepsilon \cdot (K + |A|)$$

$\square$

$\forall x, y \in \mathbb{R}$ $|x + y| \leq |x| + |y|$

$\forall \varepsilon > 0$ $\exists n_0$ $\forall n \geq n_0$ $|c_n - C| < \varepsilon$

$\lim c_n = C$

(iii) $\lim a_n = A$, $\lim b_n = B$ & $b_n \neq 0$ & $B \neq 0$ $\Rightarrow \lim \frac{a_n}{b_n} = \frac{A}{B}$

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \frac{|a_n \cdot B - b_n \cdot A|}{|b_n| \cdot |B|} = \frac{|a_n \cdot B - A \cdot B + A \cdot B - b_n \cdot A|}{|b_n| \cdot |B|} \leq$$

$$\leq \frac{|a_n \cdot B - A \cdot B|}{|b_n| \cdot |B|} + \frac{|A \cdot B - b_n \cdot A|}{|b_n| \cdot |B|} = \frac{|a_n - A| \cdot \cancel{|B|}}{|b_n| \cdot \cancel{|B|}} + \frac{|A| \cdot |B - b_n|}{|b_n| \cdot |B|}$$

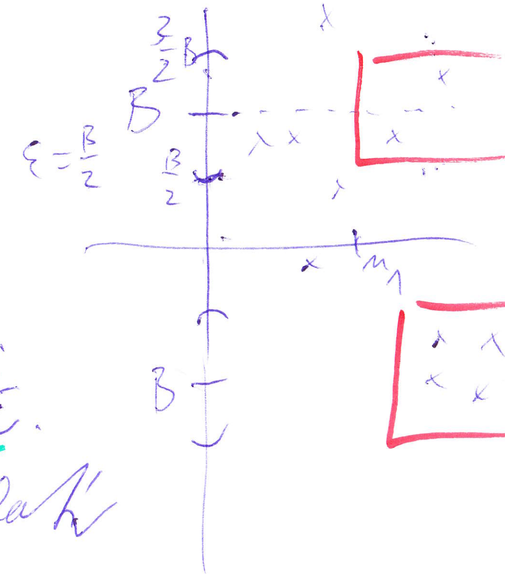
~~Nech $\varepsilon > 0$~~. K $\varepsilon_1 = \frac{|B|}{2}$ a $\lim\limits_{n\to\infty} b_n = B$

$\exists n_1 \in \mathbb{N}\ \forall n > n_1 : \quad |b_n - B| < \varepsilon_1 = \frac{|B|}{2}$

$\Rightarrow |b_n| > \frac{|B|}{2} \Rightarrow \frac{1}{|b_n|} < \frac{2}{|B|}$

$b_n \quad \{|x| < \frac{|B|}{2}\} \quad b_n$

$B \quad 0 \quad B$



Nech $\varepsilon > 0$. Z $\lim a_n = A$ $\exists n_A\ \forall n > n_A$: $|a_n - A| < \varepsilon$.

Z $\lim\limits_{n\to\infty} b_n = B$ $\exists n_B\ \forall n > n_B$: $|b_n - B| < \varepsilon$.

Položme $n_0 = \max\{n_A, n_B, n_1\}$. Potom $\forall n > n_0$ platí

$$\left|\frac{a_n}{b_n} - \frac{A}{B}\right| = \frac{|a_n \cdot B - AB + A \cdot B - b_n \cdot A|}{|b_n| \cdot |B|} \leq \frac{|a_n \cdot B - AB|}{|b_n| \cdot |B|} + \frac{|A \cdot B - b_n \cdot A|}{|b_n| \cdot |B|}$$

$$\leq \frac{|a_n - A| \cdot \cancel{|B|}}{|b_n| \cdot \cancel{|B|}} + \frac{|A| \cdot |B - b_n|}{|b_n| \cdot |B|} < \varepsilon \cdot \frac{2}{|B|} + \frac{|A| \cdot \varepsilon}{|B|} \frac{2}{|B|} = \varepsilon \cdot \left(\frac{2}{|B|} + \frac{2|A|}{|B|^2}\right)$$

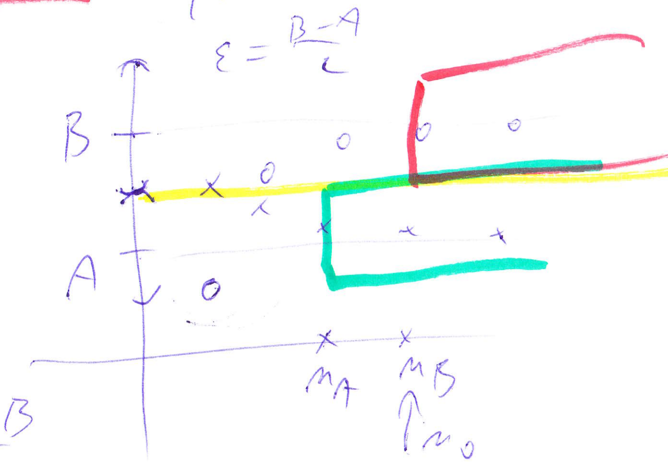
$\square$

Věta L 2.5 (limita a uspořádání)

nechť $\lim_{n\to\infty} a_n = A \in \mathbb{R}$, $\lim_{n\to\infty} b_n = B \in \mathbb{R}$.

(i) jestliže $A<B$, pak existuje $n_0 \in \mathbb{N}$, $\forall n \geq n_0$ $a_n < b_n$

(ii) jestliže $\exists n_0 \in \mathbb{N}$ tak, že $\forall n \geq n_0$ platí $a_n \geq b_n$, pak $A \geq B$.



Dk: (i) Položme $\varepsilon = \frac{B-A}{2}$.

Z $\lim_{n\to\infty} a_n = A$ $\exists n_A$ $\forall n \geq n_A$: $|a_n - A| < \varepsilon$

$$\Rightarrow a_n < A + \varepsilon = A + \frac{B-A}{2} = \frac{A+B}{2}$$

Z $\lim_{n\to\infty} b_n = B$ $\exists n_B$ $\forall n \geq n_B$: $|b_n - B| < \varepsilon$

$$\Rightarrow b_n > B - \varepsilon = B - \frac{B-A}{2} = \frac{A+B}{2}$$

Zvolme $n_0 = \max\{n_A, n_B\}$. Pak $\forall n \geq n_0$ platí

$$b_n > \frac{A+B}{2} > a_n.$$

(ii) Sporem. Nechť $A<B$. Pak podle části (i) $\exists n_1$ $\forall n \geq n_1$ $a_n < b_n$

Zároveň z předpokladu $\forall n \geq n_0$ $a_n \geq b_n$

Pak pro libovolné $n \geq n_1$ a $n \geq n_0$ platí $(a_n < b_n)$ & $(a_n \geq b_n)$ ↯ □

$$\forall n \geq n_0 \quad a_n > b_n \overset{??}{\Rightarrow} A > B.$$

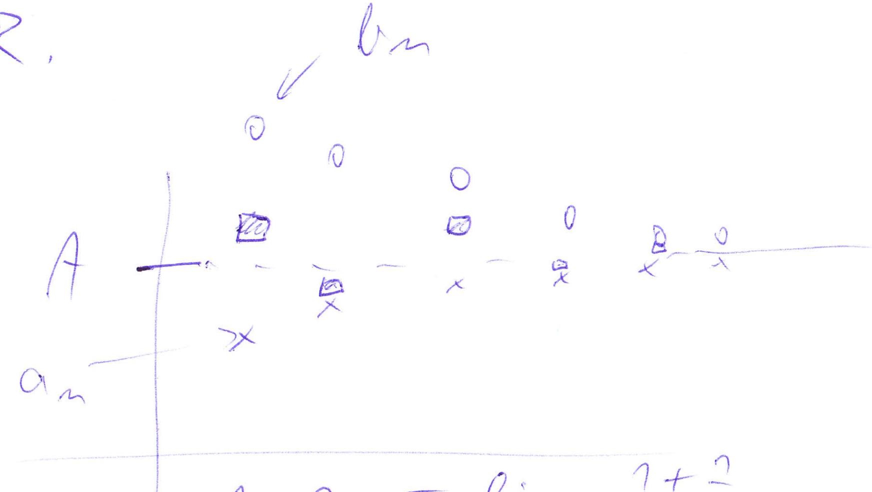
Věta L 2.6 (o dvou strážnících).

Nechť $\{a_n\}, \{b_n\}, \{c_n\}$ jsou posloupnosti splňující:

(i) $\exists n_0 \in \mathbb{N}\ \forall n \in \mathbb{N},\ n \geq n_0:\ a_n \leq c_n \leq b_n$

(ii) $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = A \in \mathbb{R}$.

Pak $\lim c_n = A$



Příklady:

a) $\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}} \stackrel{?}{=} 1$

$$\underbrace{1}_{a_n} \leq \underbrace{\sqrt{1+\frac{1}{n}}}_{c_n} \leq \underbrace{1+\frac{1}{n}}_{b_n} \qquad \lim 1 = 1 = \lim_{n\to\infty} 1+\frac{1}{n}$$

Podle věty o dvou strážnících $\lim\limits_{n\to\infty} \sqrt{1+\frac{1}{n}} = 1$

b) $\lim\limits_{n\to\infty} \sqrt[n]{n}$ $\qquad \sqrt[n]{n} = 1 + a_n \Rightarrow n = (1+a_n)^n \geq 1 + \binom{n}{2} a_n^2 = 1 + \frac{n\cdot(n-1)}{2} a_n^2$

$$\Rightarrow \frac{n-1}{\frac{n\cdot(n-1)}{2}} > a_n^2 \Rightarrow \frac{\sqrt{2}}{\sqrt{n}} > a_n$$

Víme $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Dále $\underset{\downarrow 0}{0} \leq \underset{\downarrow 0}{a_n} < \underset{\downarrow 0}{\frac{\sqrt{2}}{\sqrt{n}}}$

Podle 2 policajtů $\lim\limits_{n\to\infty} a_n = 0 \Rightarrow \lim\limits_{n\to\infty} \sqrt[n]{n} = \lim\limits_{n\to\infty} 1 + a_n = 1$

$\boxed{\forall y \geq 1:\ \sqrt{y} \leq y}$

$\lim \sqrt[n]{n} = 1$

c) $\lim\limits_{n\to\infty} \sqrt[n]{c}$

$c \in \mathbb{R},\ c > 0$

$n \geq n_0 = [c] + 1$

$\underline{c \geq 1}$ $\quad 1 \leq \sqrt[n]{c} \leq \sqrt[n]{n}$

$\downarrow n\to\infty \quad \downarrow \quad \downarrow n\to\infty$

$1 \quad 1 \quad 1$

Tedy $\lim\limits_{n\to\infty} \sqrt[n]{c} = 1$ pro $c \geq 1$ podle věty o dvou policajtech.

$\cancel{\sqrt[n]{\tfrac{1}{2}} = \tfrac{1}{\sqrt[n]{2}}}$

$0 < c < 1$: $\lim\limits_{n\to\infty} \sqrt[n]{c} = \lim\limits_{n\to\infty} \dfrac{1}{\sqrt[n]{\frac{1}{c}}} =$

$\overset{*}{=} \dfrac{\lim\limits_{n\to\infty} 1}{\lim\limits_{n\to\infty} \sqrt[n]{\frac{1}{c}}} = \dfrac{1}{1}$

$\frac{1}{c} > 1$ vím z předchozího