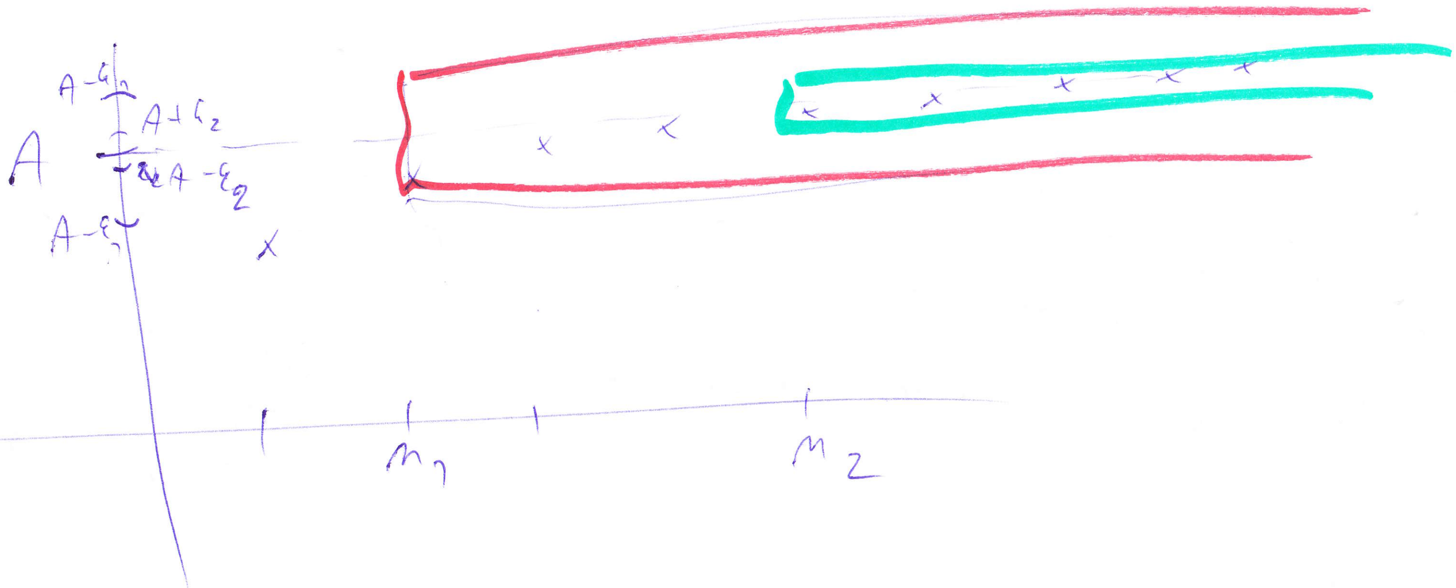
TUTOR KMA

2. Posloupnosti

$A \in \mathbb{R}$ $\quad \lim_{n\to\infty} a_n = A$.

$\forall \varepsilon > 0 \; \exists n_0 \in \mathbb{N} \; \forall n > n_0, n \in \mathbb{N} : \; |a_n - A| < \varepsilon$

$a_n \in (A-\varepsilon, A+\varepsilon)$



$\lim \frac{1}{n} = 0$

$\lim (-1)^n$ neexistuje

Příklady: 3) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$

$\forall \varepsilon > 0 \ \exists n_0 \in \mathbb{N} \ \forall n \geq n_0 : \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon$

$\iff \frac{1}{\sqrt{n}} < \varepsilon \iff \frac{1}{\varepsilon} < \sqrt{n}$

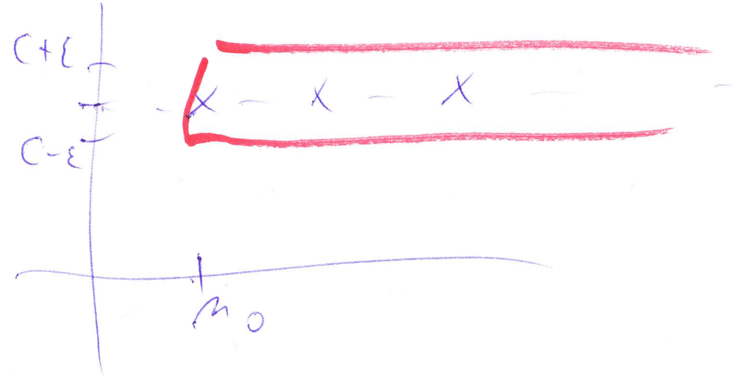
$\iff \frac{1}{\varepsilon^2} < n$

K $\varepsilon > 0$ zvolme $n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1$.

Pak $\forall n \geq n_0$ platí $\underline{n} \geq n_0 = \left[\frac{1}{\varepsilon^2}\right] + 1 > \underline{\frac{1}{\varepsilon^2}} \iff$

$\iff \frac{1}{\varepsilon} < \sqrt{n} \iff \left| \frac{1}{\sqrt{n}} - 0 \right| < \varepsilon$

4) $\lim_{n\to\infty} c = c$



$c \in \mathbb{R}$

K $\varepsilon > 0$ zvolme $n_0 = 1$.

Pak $\forall n \geq n_0 = 1$ $\quad |a_n - c| = |c - c| = 0 < \varepsilon$. ✓

5) $\lim_{n \to \infty} \sqrt[n]{n} = 1$

$$\sqrt[n]{n} = 1 + a_n \iff n = (1+a_n)^n = 1 + \binom{n}{1} \cdot a_n + \binom{n}{2} \cdot a_n^2 + \dots$$

$$\geq 1 + \frac{n \cdot (n-1)}{2} \cdot a_n^2$$

$$\Rightarrow \frac{n-1}{\frac{n \cdot (n-1)}{2}} \geq a_n^2 \Rightarrow \frac{\sqrt{2}}{\sqrt{n}} \geq a_n \geq 0$$

$$\left|\sqrt[n]{n} - 1\right| = |1 + a_n - 1| = |a_n| \leq \frac{\sqrt{2}}{\sqrt{n}} < \varepsilon$$

$$\Updownarrow$$

$$\frac{2}{n} < \varepsilon^2 \iff \frac{2}{\varepsilon^2} < n$$

K $\varepsilon > 0$ zvolme $n_0 = \left[\frac{2}{\varepsilon^2}\right] + 1$.

Pak $\forall n \geq n_0$ platí

$$\left|\sqrt[n]{n} - 1\right| = |1 + a_n - 1| = |a_n| \leq \frac{\sqrt{2}}{\sqrt{n}} < \varepsilon \quad \text{a} \quad n \geq \left[\frac{2}{\varepsilon^2}\right] + 1 > \frac{2}{\varepsilon^2}$$

Věta L2.1 (jednoznačnost vlastní limity)

Každá posloupnost má nejvýše jednu limitu.

Dk: ~~Zvolme ε~~ = Sporem. Nechť

$\lim_{n\to\infty} a_n = A$ a $\lim_{n\to\infty} a_n = B$, $A > B$.

Zvolme $\varepsilon = \frac{A-B}{3}$.

Z definice $\lim_{n\to\infty} a_n = A$ k našemu $\varepsilon = \frac{A-B}{3}$

$\exists\, n_A \in \mathbb{N}\ \forall n \geq n_A: \ |a_n - A| < \varepsilon.$

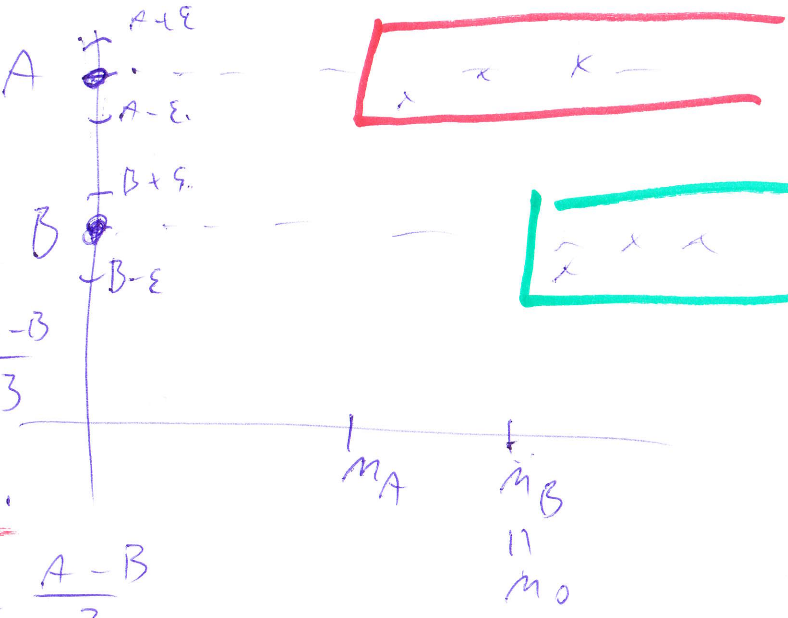
Z definice $\lim_{n\to\infty} a_n = B$ k našemu $\varepsilon = \frac{A-B}{3}$

$\exists\, n_B \in \mathbb{N}\ \forall n \geq n_B: \ |a_n - B| < \varepsilon.$

Položme $n_0 = \max\{n_A, n_B\}$. Z trojúhelníkové nerovnosti

$$|A-B| = |(A-a_{n_0}) + (a_{n_0}-B)| \leq |A-a_{n_0}| + |a_{n_0}-B| < \varepsilon + \varepsilon = \frac{2}{3}(A-B)$$

↯ □



$$|x+y| \leq |x| + |y| \quad \forall x,y \in \mathbb{R}$$

Věta L 2.2 (o omezenosti konvergentní posloupnosti)

Nechť $\{a_n\}_{n=1}^{\infty}$ má vlastní limitu. Pak je $\{a_n\}$ omezená.

Dk. Nechť $\lim_{n\to\infty} a_n = A \in \mathbb{R}$.

Položme $\varepsilon = 1$. K tomuto

$\varepsilon = 1$ $\exists n_0 \in \mathbb{N}$ $\forall n \geq n_0$

$|a_n - A| < \varepsilon \iff a_n \in (A-\varepsilon, A+\varepsilon)$

$= (A-1, A+1)$.

Množina $\{a_n : n = 1, 2, \dots, n_0\}$ je konečná,

a tedy omezená.

Položme $K = \max\{|a_1|, |a_2|, \dots, |a_{n_0}|, |A|+1\}$.

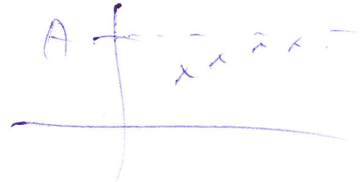
$\forall A \in \mathbb{R}$ $(A-1, A+1) \subset (-|A|-1, |A|+1)$

Pak $\forall n \in \mathbb{N}$

buď $n \leq n_0$ … $|a_n| \leq \max\{|a_1|, \dots, |a_{n_0}|\} \leq K$ ✓

nebo $n \geq n_0$ $a_n \in (A-1, A+1) \Rightarrow |a_n| \leq |A|+1 \leq K$ □

Dotaz: $\exists \lim_{n\to\infty} a_n = A \in \mathbb{R} \overset{??}{\Longrightarrow} \exists n_0 \ \forall n \geq n_0$ a_n je neklesající, nebo a_n je nerostoucí



$M \subset \mathbb{R}$ omezená $\exists K \in \mathbb{R}$ $\forall x \in M$ $|x| \leq K$

$-K \leq x \leq K$

Def Řekneme, že posloupnost $\{b_n\}_{n=1}^{\infty}$ je vybraná z posloupnosti $\{a_n\}_{n=1}^{\infty}$, jestliže existuje rostoucí posloupnost přirozených ~~čísel~~ čísel $\{n_k\}_{k=1}^{\infty}$ tak, že $b_k = a_{n_k}$. 4-6

<u>Příklad:</u>

a) posloupnost $\frac{1}{2n}$ je vybraná z $\frac{1}{n}$

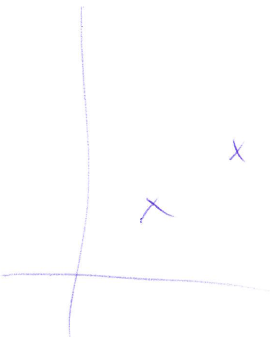
$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6} \dots$

$\frac{1}{2}, \frac{1}{4}, \frac{1}{6}$

$\frac{1}{n}$

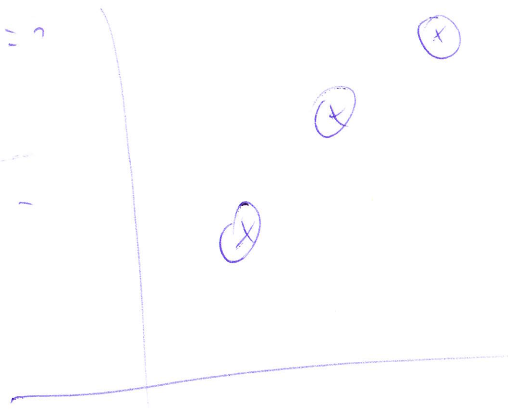
$\frac{1}{2n}$

b) posloupnost $\{2^n\}_{n=1}^{\infty}$ je vybraná z $\{n\}_{n=1}^{\infty}$



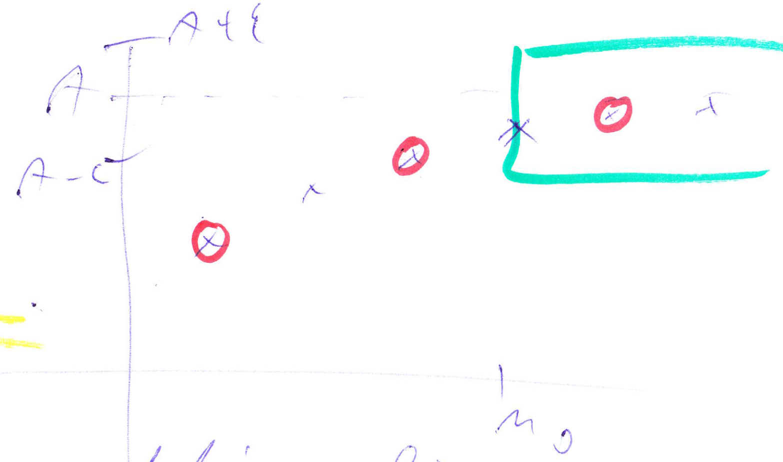
$1, 2, 3, 4, 5, 6, 7, 8, 9, \dots$

$2, 4, 8 \dots$



Věta [2.3] (o limitě vybrané posloupnosti)

Nechť $\lim_{n\to\infty} a_n = A \in \mathbb{R}$ a nechť $\{b_k\}_{k=1}^{\infty}$ je vybraná z $\{a_n\}_{n=1}^{\infty}$. Pak $\lim_{k\to\infty} b_k = A$.



Dk: Víme $\lim_{n\to\infty} a_n = A$.

$\forall \varepsilon > 0\ \exists n_0 \in \mathbb{N}\ \forall n \geq n_0 : |a_n - A| < \varepsilon$.

Chceme dokázat $\lim_{k\to\infty} b_k = A$.

$\forall \varepsilon > 0$ volme $k_0 = n_0$, kde n_0 je z definice $\lim a_n$

Nechť $k \geq k_0$. Pak $n_k \geq k \geq k_0 = n_0$.

Tedy $|b_k - A| = |a_{n_k} - A| < \varepsilon$

$b_1 = a_1 = a_{n_1}$
$b_2 = a_3 = a_{n_2}$
$b_3 = a_5 = a_{n_3}$

$\square$

Příklad: $\lim_{n\to\infty} (-1)^n$ neexistuje.

sporem $\lim_{n\to\infty} (-1)^n = A \Rightarrow \lim_{n\to\infty} (-1)^{2n} = \lim_{n\to\infty} 1 = 1 \overset{V2.3}{=} A$ — limita vybrané posloupnosti

$\Rightarrow \lim_{n\to\infty} (-1)^{2n+1} = \lim_{n\to\infty} -1 = -1$

V2.3 A ε $\square$ | $b_k = a_{n_k}$

Chceme $\forall \varepsilon > 0\ \exists k_0 \in \mathbb{N}$
$\forall k \geq k_0 : |b_k - A| < \varepsilon$.

$\boxed{\text{Věta T 2.4}}$ (aritmetika limit)

Nechť $\lim\limits_{n\to\infty} a_n = A \in \mathbb{R}$ a $\lim\limits_{n\to\infty} b_n = B \in \mathbb{R}$. Pak platí

(i) $\lim\limits_{n\to\infty} a_n + b_n = A + B$.

(ii) $\lim\limits_{n\to\infty} a_n \cdot b_n = A \cdot B$.

(iii) pokud $b_n \neq 0 \ \forall n \in \mathbb{N}$ a $B \neq 0$, pak $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \frac{A}{B}$.

Příklad: a) $\lim\limits_{n\to\infty} \frac{1}{n^2} = \lim\limits_{n\to\infty} \frac{1}{n}\cdot\frac{1}{n} \overset{\cancel{?}}{=} \lim\limits_{n\to\infty} \frac{1}{n} \cdot \lim\limits_{n\to\infty} \frac{1}{n} = 0 \cdot 0$

b) $\lim\limits_{n\to\infty} \frac{n+1}{n+2} \overset{??}{=} \cancel{\frac{\lim\limits_{n\to\infty} n+1}{\lim\limits_{n\to\infty} n+2}}$

$$\lim_{n\to\infty} \frac{n+1}{n+2} = \lim_{n\to\infty} \frac{(n+1)\cdot\frac{1}{n}}{(n+2)\cdot\frac{1}{n}} = \lim_{n\to\infty} \frac{1+\frac{1}{n}}{1+\frac{2}{n}} =$$

$$\overset{\cancel{?}}{=} \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(1+\frac{2}{n}\right)} \overset{\cancel{?}}{=} \frac{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{1}{n}}{\lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} 2 \cdot \lim\limits_{n\to\infty} \frac{1}{n}} = \frac{1+0}{1+2\cdot 0} = 1$$

c) obecně neplatí $\lim\limits_{n\to\infty} (a_n + b_n) = \lim\limits_{n\to\infty} a_n + \lim\limits_{n\to\infty} b_n$

$$0 = \lim_{n\to\infty} 0 = \lim_{n\to\infty} (-1)^n + (-1)^{n+1} \overset{??}{=} \lim_{n\to\infty} (-1)^n + \lim_{n\to\infty} (-1)^{n+1}$$