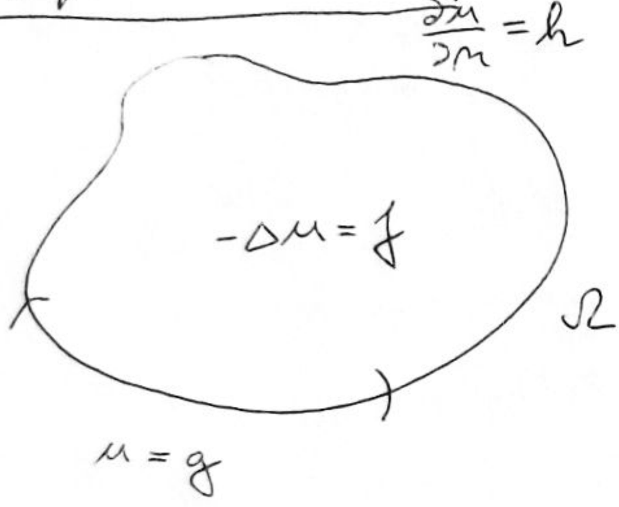
$\boxed{\text{Poissonova - Laplaceova rce}}$



- f, g, h dané funkce
- u neznámá funkce

$\boxed{v\ \mathbb{R}^n}$ Odvození fundamentálního řešení v $\mathbb{R}^3$

Provedeme FT: ozn. $V = \hat{u}$

$$\widehat{-\Delta u} = \hat{f} \longrightarrow 4\pi^2 |\xi|^2 V(\xi) = \hat{f}(\xi)$$

$$\longrightarrow V(\xi) = \hat{f}(\xi) \cdot \frac{1}{4\pi^2|\xi|^2} \longrightarrow u(x) = \left(\hat{f}(\xi) \cdot \frac{1}{4\pi^2|\xi|^2}\right)^{\vee}(x)$$

$$u(x) = f * \mathcal{U}, \text{ kde } \hat{\mathcal{U}}(\xi) = \frac{1}{4\pi^2|\xi|^2}.$$

$$\mathcal{U}(x) \text{ "=" } \int_{\mathbb{R}^3} \frac{1}{4\pi^2|\xi|^2} e^{2\pi i(\xi \cdot x)} \, d\xi = \int_{\mathbb{R}^3} \frac{1}{4\pi^2|\xi|^2} e^{2\pi i |x| \xi_3} \, d\xi$$

$\parallel$

$\mathcal{U}(|x| e_3)$

$\quad \hookrightarrow (0, 0, |x|)$

sférické souřadnice Φ: $\xi_1 = r \cos\alpha \cos\beta$

$\xi_2 = r \cos\alpha \sin\beta$

$\xi_3 = r \sin\alpha$

$\operatorname{Jac}\Phi = r^2 \cos\alpha$;

$r \in (0, +\infty),\ \alpha \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ \beta \in (0, 2\pi)$

$$= \int_0^{+\infty}\int_0^{2\pi}\int_{-\pi/2}^{\pi/2} \frac{1}{4\pi^2 r^2} e^{2\pi i |x| r \sin\alpha} r^2 \cos\alpha \, d\alpha \, d\beta \, dr = \frac{1}{2\pi} \int_0^{+\infty}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{2\pi i |x| r \sin\alpha} \frac{2\pi i |x| r \cos\alpha}{2\pi i |x| r} \, d\alpha \, dr$$

$$= \frac{1}{2\pi|x|}\int_0^{+\infty} \frac{1}{2\pi i r}\left[e^{2\pi i |x| r \sin\alpha}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dr =$$

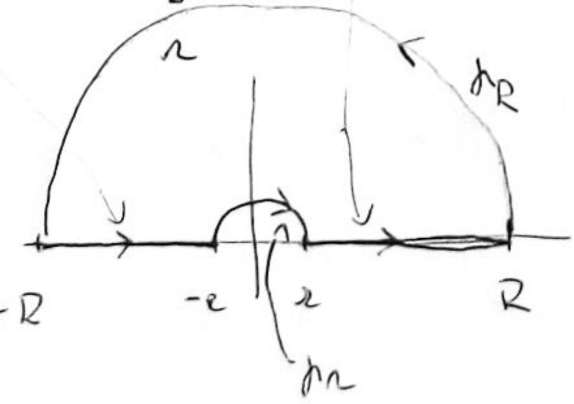
$$= \frac{1}{2\pi^2|x|}\int_0^{+\infty}\frac{1}{r}\underbrace{\frac{1}{2i}\left(e^{2\pi|x|ri} - e^{-2\pi|x|ri}\right)}_{=\sin(2\pi|x|r)} dr$$

$$\overset{KA}{=} \frac{1}{2\pi^2|x|}\lim_{\substack{R\to+\infty\\ r\to 0+}}\frac{1}{2}\left(\int_{-R}^{-r}\frac{\sin(2\pi|x|r)}{r}\,dr + \int_r^R \frac{\sin 2\pi|x|r}{r}\,dr\right)$$

$$= \frac{1}{2\pi^2|x|}\lim_{\substack{R\to+\infty\\ r\to 0+}}\frac{1}{2}\frac{1}{i}\left(\int_{-R}^{-r}\frac{e^{2\pi i|x|r}}{r}\,dr + \int_r^R \frac{e^{2\pi i|x|r}}{r}\,dr\right) = \mathcal{U}(x)$$

$$F(z) = \frac{e^{2\pi i|x|z}}{z}$$

γ :



KA :

$$\int_{\gamma_R} F(z)\,dz \xrightarrow{R\to+\infty} 0$$

$$\int_{\gamma_r} F(z)\,dz \xrightarrow{r\to 0+} -\frac{1}{2}\,2\pi i\,\mathrm{res}_0 F = -\pi i$$

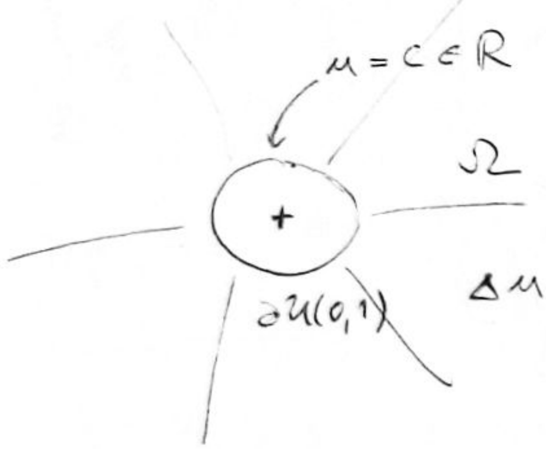
$$\mathcal{U}(x) = \frac{1}{2\pi^2|x|}\,\frac{\pi i}{2i} = \frac{1}{4\pi|x|} \quad ✓$$

$m-2 = 1$ $(m=3)$

$\varkappa_m = \varkappa_3 = 4\pi$

K Větě 5.7:

(3)

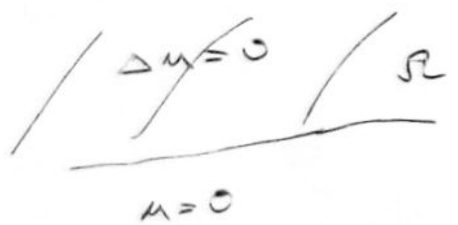


$u(x) \equiv c$ na Ω je řešení

$u(x) := \frac{c}{|x|^{n-2}}$ na Ω je řešení

Dcv: $\Delta\left(\frac{1}{|x|^{n-2}}\right) = 0$ v $\mathbb{R}^n \setminus \{0\}$

(2)



$$\left.\begin{aligned} u(x,y) &:= 0 \\ u(x,y) &:= y \end{aligned}\right\} \text{ jsou řešení.}$$

$\boxed{\text{K Větě 5.8}}$

Proveď FT v x. $V = \mathcal{F}_x u$

$V(\xi, y)$

$\partial_{xx} u + \partial_{yy} u = 0 \qquad \mathcal{F}_x$

$$\left.\begin{aligned} \partial_{yy} V - 4\xi^2\pi^2 V &= 0 \\ V(\cdot, 0) &= \hat{g} \end{aligned}\right]$$

f. s.: $\lambda^2 - 4\pi\xi^2 = 0$, vl. č.: $\lambda = \pm 2\pi\xi$

$e^{2\pi y \xi}, \; e^{-2\pi y \xi}$

$V(\xi, y) = \hat{g}(\xi) \cdot e^{-2\pi y |\xi|}$

$u(x,y) = \left(\mathcal{F}^{-1}_{x,\xi} V\right)(x,y) =$

$= (g * u_0)(x,y)$

$(\mathcal{F}_x u_0)(\xi, y) = e^{-2\pi y |\xi|}$

$$u_0(x,y) = \int_{\mathbb{R}} e^{-2\pi y|\xi|} e^{2\pi i x \xi} \, d\xi =$$

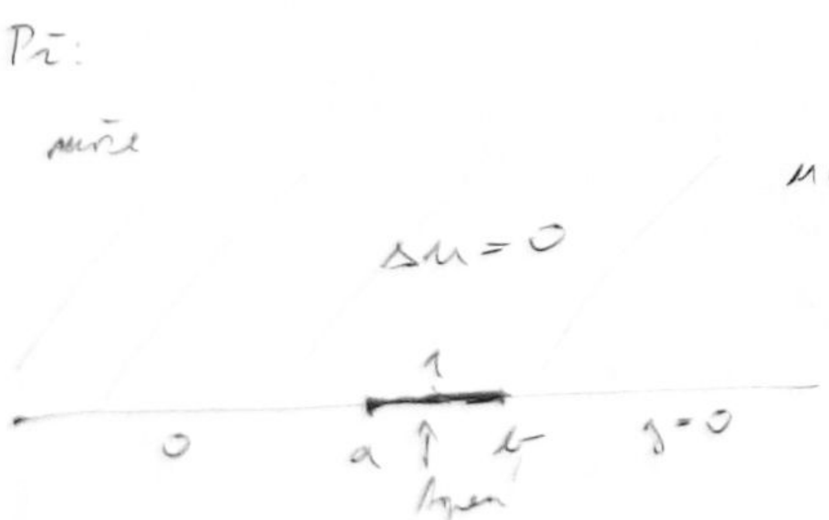
$$\int_{-\infty}^{0} e^{\underbrace{2\pi y \xi + 2\pi i x \xi}_{\xi 2\pi(y+ix)}} d\xi + \int_{0}^{+\infty} e^{\underbrace{-2\pi y \xi + 2\pi i x \xi}_{\xi 2\pi(-y+ix)}} d\xi =$$

$$\left[\frac{e^{2\pi\xi(y+ix)}}{2\pi(y+ix)} \right]_{-\infty}^{0} + \left[\frac{e^{2\pi\xi(-y+ix)}}{2\pi(-y+ix)} \right]_{0}^{+\infty} =$$

$$\frac{1}{2\pi(y+ix)} \pm \frac{1}{2\pi(\pm y \mp ix)} = \frac{y - ix + y + ix}{2\pi(y^2+x^2)} =$$

$$= u_0(x,y) = \frac{y}{\pi(y^2+x^2)}$$

Př:



$$u(x,y) = \frac{y}{\pi} \int_a^b \frac{1}{(x-t)^2+y^2} \, dt$$

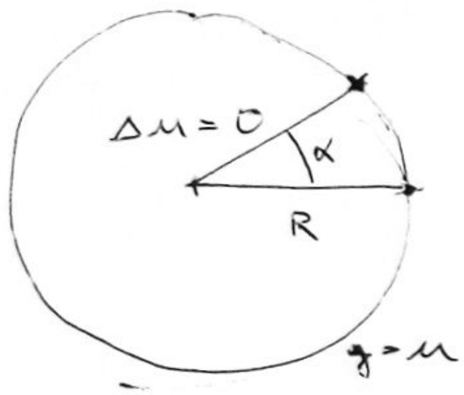
$$= \frac{1}{\pi y} \int_a^b \frac{1}{\left(\frac{x-t}{y}\right)^2 + 1} \, dt$$

$$\frac{t-x}{y}$$

počáteční stav

$$= \frac{1}{\pi} \left[\operatorname{arctg}\left(\frac{t-x}{y}\right) \right]_{t=a}^{b} = \frac{1}{\pi}\left(\operatorname{arctg}\frac{b-x}{y} - \operatorname{arctg}\frac{a-x}{y} \right)$$

<u>V 5.9</u>



g zadávána úhlem α

- je 2π per.

→ Fourierova řada g

u hledáme $u(r,\varphi)$ —

$r \in (0,R)$

$\varphi \in (0,2\pi)$

okrajová podm.: $u(R,\varphi) = g(\varphi)$

$$\longrightarrow u(r,\varphi) = \sum_{n\in\mathbb{Z}} a_n \left(\frac{r}{R}\right)^{|n|} e^{in\varphi}, \text{ kde}$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\, e^{-in\alpha}\, d\alpha .$$

$\boxed{\text{KV5.10}}$

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\left[\sum_{n=-\infty}^{-1}\left(\frac{r}{R}\right)^{-n} e^{in(\varphi-\alpha)} + \overbrace{\sum_{n=0}^{+\infty}\left(\frac{r}{R}\right)^{n} e^{in(\varphi-\alpha)}}^{\left(\frac{r}{R}e^{i(\varphi-\alpha)}\right)^n}\right] d\alpha$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\left(\underbrace{\sum_{m=1}^{+\infty}\left(\frac{r}{R}\right)^{m} e^{-i(\varphi-\alpha)m}}_{\left(\frac{r}{R}e^{i(\alpha-\varphi)}\right)^m} + \frac{1}{1-\frac{r}{R}e^{i(\varphi-\alpha)}}\right) d\alpha$$

$m = -n$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\left(\frac{\frac{r}{R}e^{i(\alpha-\varphi)}}{1-\frac{r}{R}e^{i(\alpha-\varphi)}} + \frac{1}{1-\frac{r}{R}e^{i(\varphi-\alpha)}}\right) d\alpha$$

$$(\quad) = \frac{\cancel{\frac{r}{R} e^{i(\alpha-\varphi)}} - \frac{r^2}{R^2} + 1 - \cancel{\frac{r}{R} e^{i(\alpha-\varphi)}}}{1 - \frac{r}{R}\left(e^{i(\alpha-\varphi)} + e^{i(\varphi-\alpha)}\right) + \left(\frac{r}{R}\right)^2}$$

$$= \frac{R^2 - r^2}{R^2 - 2Rr\cos(\alpha-\varphi) + r^2}$$

$$u(r,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} g(\alpha)\frac{R^2-r^2}{R^2-2Rr\cos(\alpha-\varphi)+r^2}\,d\alpha$$

$$\boxed{\text{Veta } 5.14 \Rightarrow \text{Veta } 5.10}$$

$$u(\overbrace{r\cos\alpha, r\sin\alpha}^{x}) \overset{V5.14}{=} \frac{1}{2\pi \cancel{R}}\int_0^{2\pi} g(R\cos t, R\sin t)\frac{R^2-r^2}{(r\cos\alpha - R\cos t)^2 + (r\sin\alpha - R\sin t)^2}\,\cancel{R}\,dt$$

$$\gamma: t \longrightarrow \begin{pmatrix} R\cos t \\ R\sin t \end{pmatrix} \quad \alpha \in (0, 2\pi)$$

$$ds = R\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(R\cos t, R\sin t)\frac{R^2-r^2}{r^2+R^2-2rR\underbrace{(\cos\alpha\cos t + \sin\alpha\sin t)}_{\cos(\alpha-t)}}\,dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} g(R\cos t, R\sin t)\frac{R^2-r^2}{r^2+R^2-2rR\cos(\alpha-t)}\,dt$$