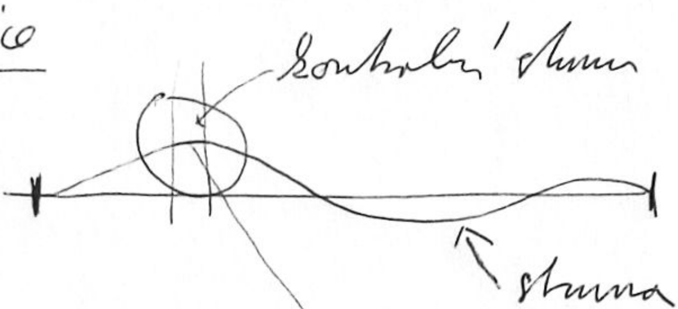
Odvození vlnové rovnice

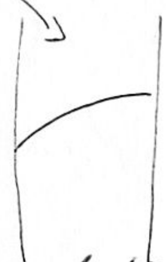
pro strunu:



2. Newtonův zákon – bilance hybnosti

změna hybnosti = síla

→ použití na kontrolní objem



$u(t,x)$ ... popisuje výchylku struny v bodě x v čase t

$\partial_t u(t,x)$ ... rychlost struny

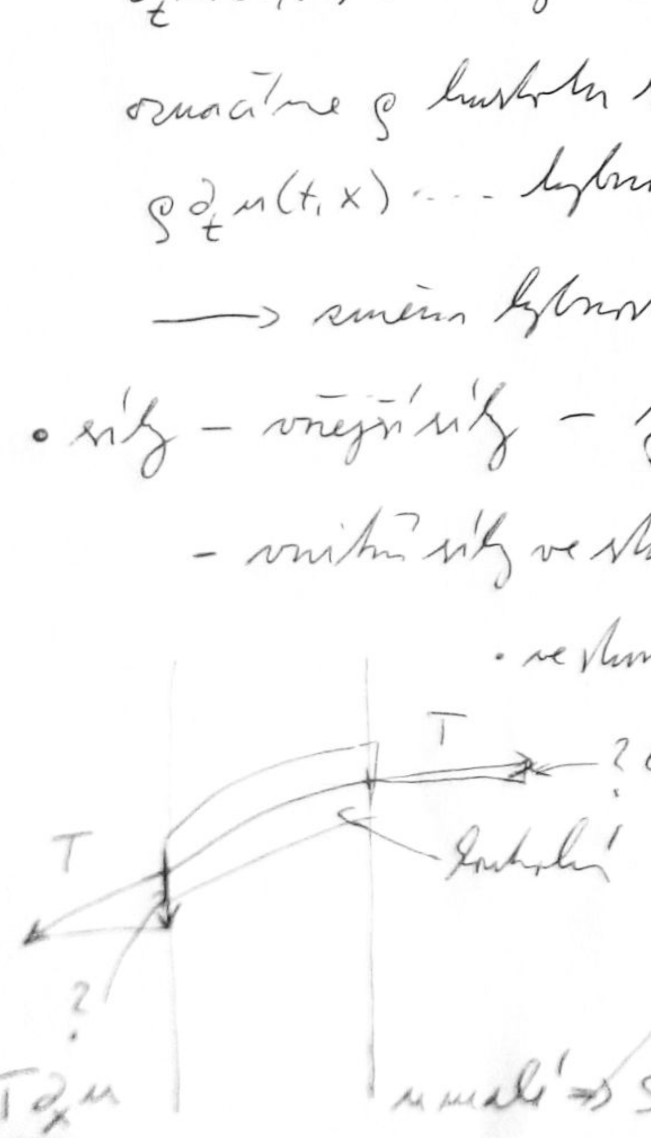
označíme ρ hustotu struny

$\rho\, \partial_t u(t,x)$ ... hybnost

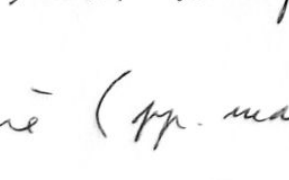
je-li ρ konstanta

→ změna hybnosti: $\partial_t\left[\rho\, \partial_t u(t,x)\right] = \rho\, \partial_t^2 u(t,x)$

- síly – vnější síly – f hustota vnějších sil
  - vnitřní síly ve struně (pp. malé u)
    - ve struně je napětí T



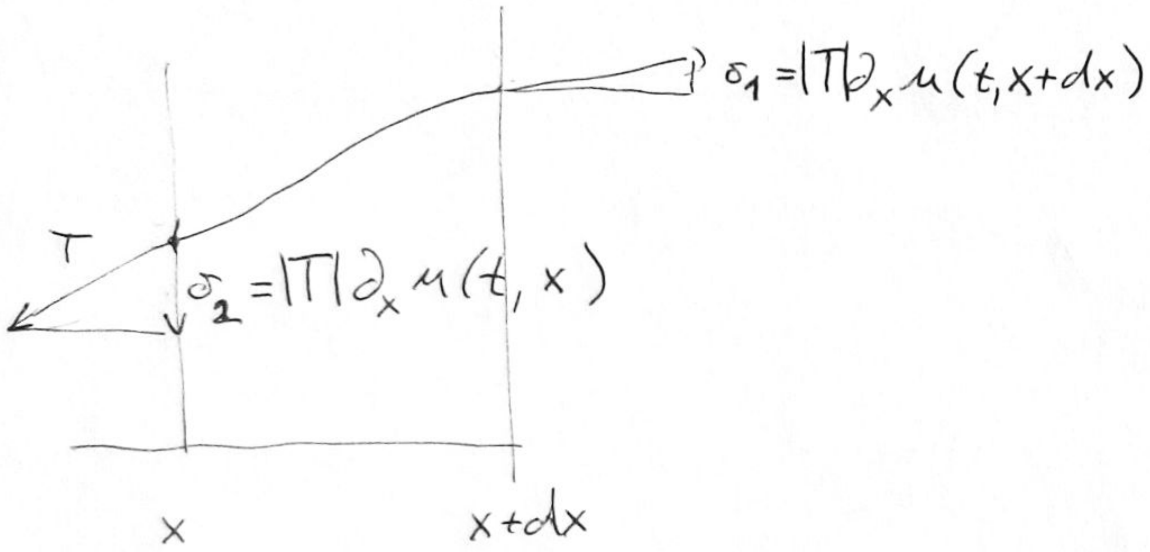
? $\sigma_1 = T\partial_x u$; T ... ve směru tečny k grafu u



$\sin\theta = \frac{\sigma_1}{|T|}$

$\operatorname{tg}\theta = \partial_x u = \frac{\sigma_1}{S}$

u malé $\Rightarrow S \doteq T \Rightarrow \sigma_1 \doteq T\partial_x u$



vnitřní síly: $\sigma_1 - \sigma_2$ (promítnuté ve směru osy y)

bilance hybnosti pro úsek struny v $[x, x + \underbrace{dx}_{\substack{>0 \\ \text{malé}}}]$

$\rho > 0$ konstantní

$$\rho\, \partial_t^2 u(t,x) \cdot dx = f(t,x)\, dx + |T| \left(\partial_x u(t, x+dx) - \partial_x u(t,x) \right)$$

$$\rho\, \partial_t^2 u(t,x) = f(t,x) + |T| \frac{\partial_x u(t,x+dx) - \partial_x u(t,x)}{dx}$$

$$dx \to 0+ : \quad \rho\, \partial_t^2 u(t,x) = f(t,x) + |T|\, \partial_x^2 u(t,x)$$

$$\longrightarrow \boxed{\rho\, \partial_t^2 u(t,x) - |T| \underbrace{\partial_x^2 u(t,x)}_{\Delta u \text{ v } 1D} = f(t,x)}$$

ρ, $|T|$ nezáporné parametry struny

Řešení Cauchyovy úlohy pro vlnovou rovnici

$$\frac{1}{c^2}\partial_t^2 u - \Delta u = f \qquad v\ (t,x)\in(0,+\infty)\times\mathbb{R}^n$$

$$\left.\begin{aligned} u(0,x) &= g_0(x) \\ \partial_t u(0,x) &= g_1(x) \end{aligned}\right\}\ x\in\mathbb{R}^n$$

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1) FT v x: $U = \hat{u}$

$$\frac{1}{c^2}\partial_t^2 U + 4\pi^2|\xi|^2 U = \hat{f} \qquad v\ (t,\xi)\in(0,+\infty)\times\mathbb{R}^n$$

$$\left.\begin{aligned} U(0,\xi) &= \hat{g}_0(\xi) \\ \partial_t U(0,\xi) &= \hat{g}_1(\xi) \end{aligned}\right\}\ \xi\in\mathbb{R}^n$$

2) Řešíme ODR v t: charakter. rce: $\frac{1}{c^2}\lambda^2 + 4\pi^2|\xi|^2 = 0$
ξ pevné

$$\lambda_{1,2} = \pm 2\pi|\xi|c\,i$$

f.s.: $\sin(2\pi|\xi|ct)$, $\cos(2\pi|\xi|ct)$

OŘ homogenní rce: $A\sin(2\pi|\xi|ct) + B\cos(2\pi|\xi|ct)$

Řeš. hom. rce, které splňuje poč. podm.: $B = \hat{g}_0(\xi)$

$$A\cdot 2\pi|\xi|c = \hat{g}_1(\xi)$$

$$\longrightarrow \frac{\hat{g}_1(\xi)}{2\pi|\xi|c}\sin(2\pi|\xi|ct) + \hat{g}_0(\xi)\cos(2\pi|\xi|ct)$$

$$= \hat{g}_1(\xi)\cdot\frac{\sin(2\pi|\xi|ct)}{2\pi|\xi|c} + \hat{g}_0(\xi)\,\partial_t\left(\frac{\sin 2\pi|\xi|ct}{2\pi|\xi|c}\right)$$

Řeš. rce s PS s nulovými poč. podm.

pomocí variace konstant: $A(t)\sin(2\pi|\xi|ct) + B(t)\cos(2\pi|\xi|ct)$

$$A'(t)\sin(2\pi|\xi|ct) + B'(t)\cos(2\pi|\xi|ct) = 0 \qquad \begin{array}{l} /2\pi|\xi|c\sin() \\ /2\pi|\xi|c\cos() \end{array}$$

$$A'(t)\,2\pi|\xi|c\cos(2\pi|\xi|ct) - B'(t)\,2\pi|\xi|c\sin(2\pi|\xi|ct) = c^2\hat{f}(t,\xi)$$

$$A'(t)2\pi|\xi|c = c^2\hat{f}(t,\xi)\cdot\cos(2\pi|\xi|ct) \qquad \begin{array}{l} /\cdot\cos \\ /-\sin \end{array}$$

$$A'(t) = c^2\hat{f}(t,\xi)\,\frac{\cos(2\pi|\xi|ct)}{2\pi|\xi|c}$$

$$B'(t)\,2\pi|\xi|c = -c^2\hat{f}(t,\xi)\,\sin(2\pi|\xi|c)$$

$$B'(t) = -c^2\hat{f}(t,\xi)\,\frac{\sin(2\pi|\xi|c)}{2\pi|\xi|c}$$

$$\longrightarrow \int_0^t c^2\hat{f}(s,\xi)\,\frac{\cos(2\pi|\xi|cs)}{2\pi\xi c}\,ds\,\sin(2\pi|\xi|ct) +$$

$$-\int_0^t c^2\hat{f}(s,\xi)\,\frac{\sin(2\pi|\xi|cs)}{2\pi|\xi|c}\,ds\,\cos(2\pi|\xi|ct)$$

$$= \int_0^t c\,\frac{\hat{f}(s,\xi)}{2\pi|\xi|}\left(\cos(2\pi|\xi|cs)\cdot\sin(2\pi|\xi|ct) - \sin(2\pi|\xi|cs)\cos(2\pi|\xi|ct)\right)ds$$

$$= \int_0^t c^2\hat{f}(s,\xi)\,\frac{\sin(2\pi|\xi|(t-s))}{2\pi|\xi|c}\,ds$$ ... má nulové poč. podm.

$$u(t,\xi) = \overbrace{\hat{g}_1(\xi)\cdot\frac{\sin(2\pi|\xi|ct)}{2\pi|\xi|c}}^{(PS)_1} + \overbrace{\hat{g}_0(\xi)\cdot\partial_t\left(\frac{\sin 2\pi|\xi|ct}{2\pi|\xi|c}\right)}^{(PS)_2}$$

$$\underbrace{+\int_0^t c^2\hat{f}(s,\xi)\,\frac{\sin[2\pi|\xi|(t-s)]}{2\pi|\xi|c}\,ds}_{(PS)_3}$$

3) Provedeme $(FT)^{-1}$: Ať $W(t,x)$ je taková, že

$$(\mathcal{F}_x W)(t,\xi) = \frac{\sin 2\pi|\xi|ct}{2\pi|\xi|c}$$

$$t>0,\ x,\xi\in\mathbb{R}^n$$

$$(PS)_1^{\vee} = g_1 * W$$

$$(PS)_2^{\vee} = g_0 * \partial_t(W) = \partial_t(g_0 * W)$$

$$(PS)_3^{\vee} = \int_0^t c^2 f(s,\cdot) *_x W(t-s,\cdot)\,ds$$

$$= c^2 fY \underset{t,x}{*} WY$$

$$\longrightarrow \boxed{u = g_1 *_x W + \partial_t(g_0 *_x W) + c fY \underset{t,x}{*} WY}$$

na $(0,+\infty)\times\mathbb{R}^n$

$\mathbb{R}^1$ – $(FT)^{-1}$ z $\dfrac{\sin(2\pi\xi ct)}{2\pi\xi c}$ $\qquad \xi, x \in \mathbb{R}$

$\boxed{\text{Fundamentální řešení pro } x \in \mathbb{R}^1,\ n = 1}$

$$W(t,x) \overset{(*)}{=} \int_{\mathbb{R}} \frac{\sin 2\pi\xi ct}{2\pi\xi c}\, e^{-2\pi i\xi x}\, d\xi =$$

$$\int_{\mathbb{R}} \frac{1}{4\pi i\xi c}\left(e^{2\pi i\xi ct} - e^{-2\pi i\xi ct}\right)\cdot e^{-2\pi i\xi x}\, d\xi$$

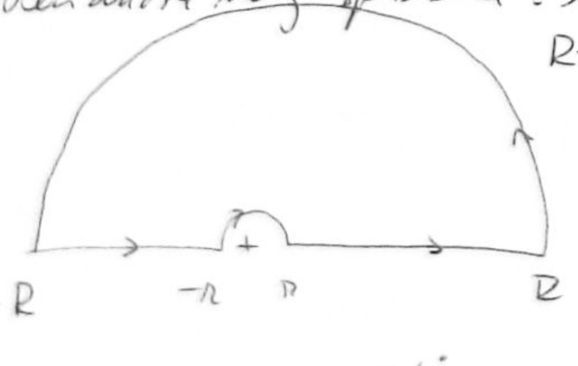
$$= \int_{\mathbb{R}} \frac{e^{2\pi i\xi(ct-x)}}{4\pi i\xi c}\, d\xi - \int_{\mathbb{R}} \frac{e^{2\pi i\xi(-ct-x)}}{4\pi i\xi c}\, d\xi$$

* $\dfrac{\sin 2\pi\xi ct}{2\pi\xi c} \notin L^1(\mathbb{R}) \Rightarrow$ integrál je třeba rozumět

Potřebujeme: $\displaystyle\int_{\mathbb{R}} \frac{e^{\alpha\xi i}}{\xi}\, d\xi$ pro $\alpha \in \mathbb{R}$

Pomocí Reziduové věty spočteme: $\displaystyle\lim_{R\to+\infty}\lim_{r\to 0+}\left(\int_{-R}^{-r} \frac{e^{\alpha\xi i}}{\xi}\, d\xi + \int_{r}^{R} \frac{e^{\alpha\xi i}}{\xi}\, d\xi\right) = I$

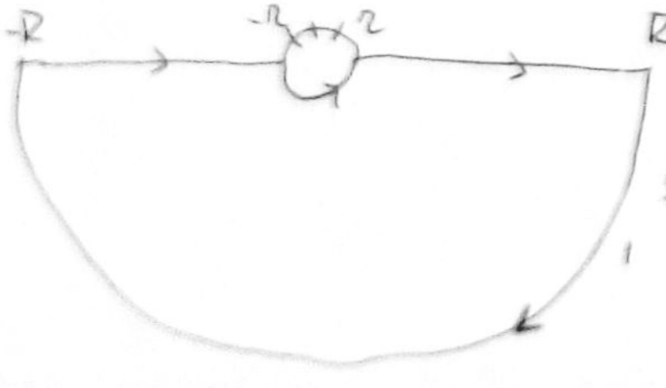
$\alpha > 0$:



$\dfrac{e^{\alpha z i}}{z}$

$$I = 2\pi i\, \frac{1}{2}\, \operatorname{res}_0 \frac{e^{z\alpha i}}{z} = \pi i$$

$\alpha < 0$

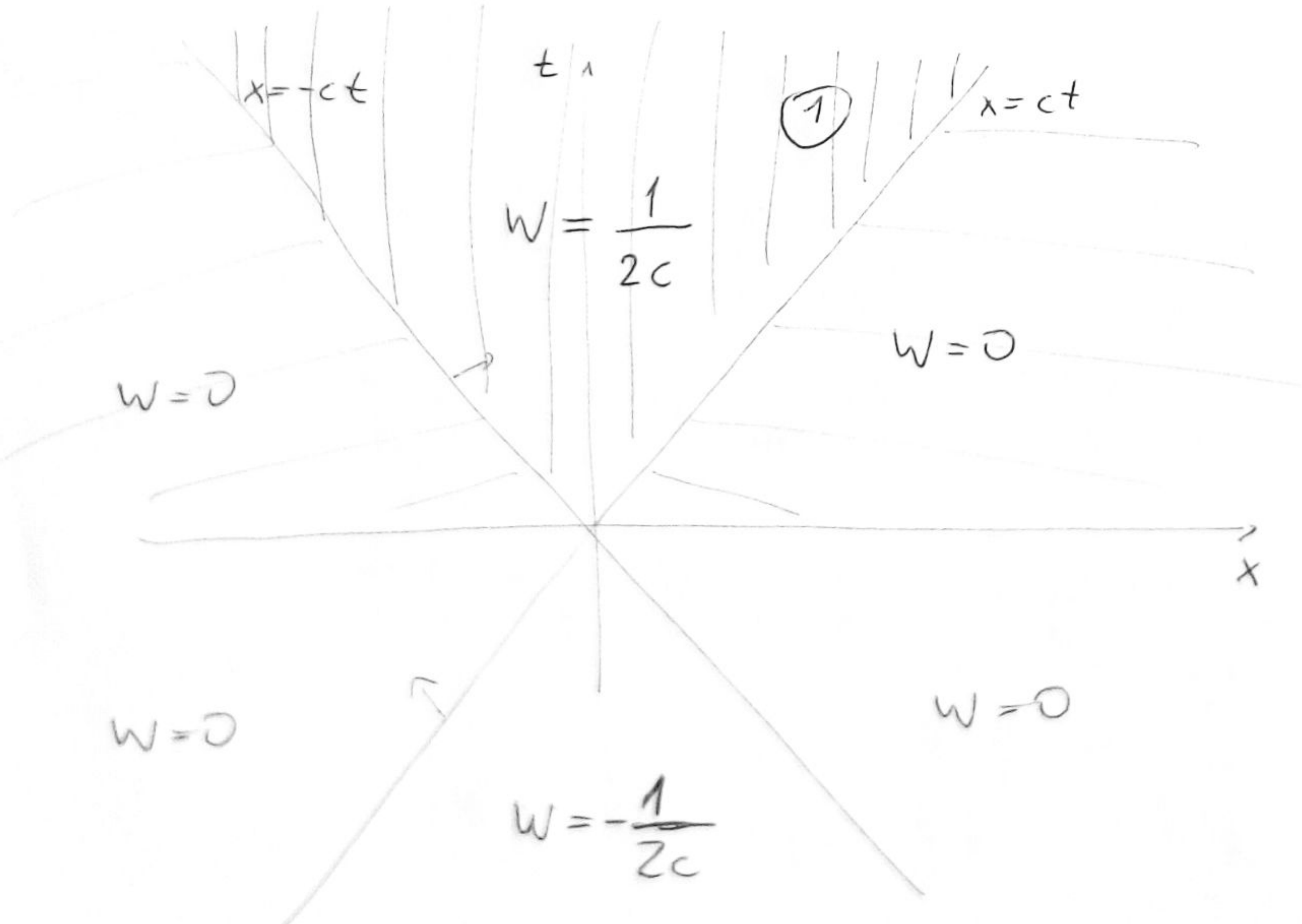


$I = -\pi i$

$\dfrac{e^{\alpha z i}}{z}$

$$W(t,x) = \frac{1}{4\pi i c}\left(\int_{\mathbb{R}} \frac{e^{2\pi(ct-x)i\xi}}{\xi}\, d\xi - \int_{\mathbb{R}} \frac{e^{2\pi(-ct-x)i\xi}}{\xi} \right)$$

$$= \begin{cases} 0 & ct-x \text{ a } -ct-x \text{ mají stejné znaménko} \\ & \text{tj. } x^2-(ct)^2>0,\ |x|>ct \\ \frac{1}{4\pi i c}(2\pi i) & ct-x>0 \\ & -ct-x<0 \quad (1) \\ -\frac{2\pi i}{4\pi i c} & ct-x<0 \\ & -ct-x>0 \end{cases}$$



$$t>0: \quad W(t,x) = \begin{cases} 0 & |x|>ct \\ \frac{1}{2c} & |x|<ct,\ x\in(-ct,ct) \end{cases}$$

$$\boxed{\text{D'Alembertův vzorec}}$$

$$g_1 *_x W(t,x) = \int_{\mathbb{R}} W(t,y)\, g_1(x-y)\, dy =$$

$$= \int_{-ct}^{ct} \frac{1}{2c}\, g_1(\underbrace{x-y}_{r})\, dy = \Big|\; x-y=r$$

$$= \frac{1}{2c} \int_{x-ct}^{x+ct} g_1(r)\, dr$$

$$\partial_t (g_0 *_x W)(t,x) = \partial_t \left(\frac{1}{2c} \int_{x-ct}^{x+ct} g_0(r)\, dr \right) = \frac{1}{2c}\left(c\, g_0(x+ct) - c\, g(x-ct) \right)$$

$$= \frac{1}{2}\left(g_0(x+ct) - g_0(x-ct) \right)$$

$$c\, fY *_{t,x} WY(t,x) = c \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} Y(s)\, \chi_{(-cs,\,cs)}(y)\, \frac{1}{2}\, f(t-s,\, x-y) \underbrace{\,}_{Y(t-s)} dy\, ds$$

$$= \frac{c}{2} \int_0^t \int_{-cs}^{cs} f(t-s,\, x-y)\, dy\, ds = \Big|\; t-s=\tau \;\Big|$$

$$= \frac{c}{2} \int_0^t \int_{-c(t-\tau)}^{c(t-\tau)} f(\tau,\, x-y)\, dy\, d\tau = \Big|\; x-y=r \;\Big|$$

$$= \frac{c}{2} \int_0^t \left(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\tau, r)\, dr \right) d\tau$$