Fourierova transformace

$f \in L^1(\mathbb{R}^m) \longrightarrow \hat{f} \in C(\mathbb{R}^m)$ omezená

$$\hat{f}(\xi) = \int_{\mathbb{R}^m} f(x) e^{-2\pi i (x, \xi)} dx$$

$$\widehat{\frac{\partial f}{\partial x_i}}(\xi) = 2\pi i \, \xi_i \hat{f}(\xi)$$

Cvičení: $y'' - y = e^{-\pi x^2}$, $m = 1$

aplikujeme FT: $Y = \hat{y}$, $F = \hat{f}$, $f(x) = e^{-\pi x^2}$

$$(2\pi i \xi)^2 Y - Y = F$$

$$-(4\pi^2 \xi^2 + 1) Y = F$$

$$Y(\xi) = \frac{-F(\xi)}{4\pi^2\xi^2 + 1}, \quad y(x) = \check{Y}(x) = \left(-\frac{F(\xi)}{4\pi^2\xi^2+1}\right)^{\vee}(x)$$

$$y(x) = -F^{\vee}(x) * \left(\frac{1}{4\pi^2\xi^2+1}\right)^{\vee}(x)$$

$$= -(f * \mathcal{U})(x), \text{ kde } \mathcal{U}(x) = \left(\frac{1}{4\pi^2\xi^2+1}\right)^{\vee}(x)$$

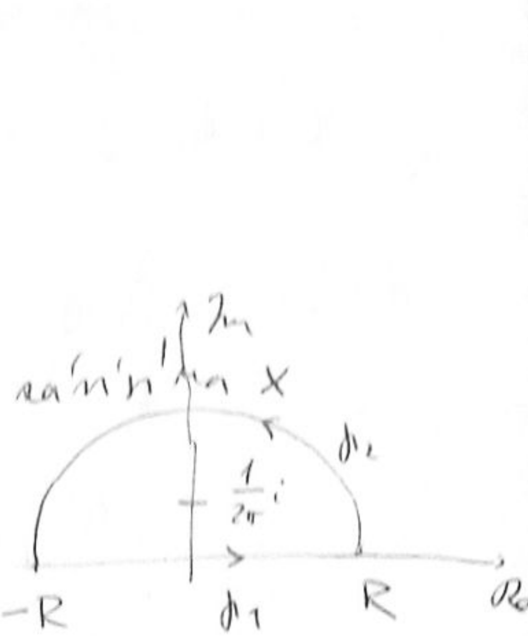
$$= -\int_{\mathbb{R}} f(x-y) \frac{e^{-|y|}}{2} dy$$

Vyjádřit $\left(\frac{1}{4\pi^2\xi^2+1}\right)^{\vee}(x)$:

$$\int_{\mathbb{R}} \frac{1}{4\pi^2\xi^2+1}\, e^{2\pi i \xi x}\, d\xi$$

Pomocí Residuové věty: γ … hranice závisí na x

$\boxed{x>0}$: γ_R



$$F(z) = \frac{e^{2\pi i z x}}{4\pi^2 z^2+1}$$

Lemma o velké polokružnici: • $\int_{\gamma_2} F(z)\,dz \xrightarrow{R\to+\infty} 0$

• $\int_{\gamma_1} F(z)\,dz \xrightarrow{R\to+\infty} \mathcal{U}(x)$

Residuová věta: $\int_{\gamma_1} F + \int_{\gamma_2} F = 2\pi i$ součet residuí uvnitř γ

Izolované sing. F: $4\pi^2 z^2+1=0$; $z^2 = -\frac{1}{4\pi^2}$

$$z = \pm \frac{1}{2\pi} i$$

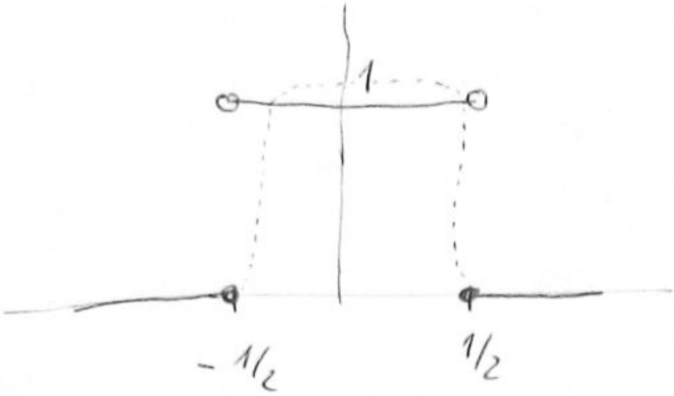
$$\operatorname{res}_{\frac{1}{2\pi}i} F(z) = \left.\frac{e^{2\pi i z x}}{8\pi^2 z}\right|_{z=\frac{1}{2\pi}i} = \frac{e^{-x}}{4\pi i}$$

$\rightarrow \mathcal{U}(x) = 2\pi i \frac{e^{-x}}{4\pi i} = \frac{e^{-x}}{2}$ pro $\underline{x>0}$

$\mathcal{U}(x) = \frac{e^{-|x|}}{2}$ pro $x\in\mathbb{R}$ ze sudosti $\mathcal{U}$.

Ke konvoluci

Př: $f(x) := \chi_{(-1/2,\, 1/2)}(x)$



$$\int_{\mathbb{R}} f = 1.$$

$$f * g(x) = \ ?$$

$$f * g(0) = \int_{\mathbb{R}} f(-y)\, g(y)\, dy = \int_{-1/2}^{1/2} g(y)\, dy$$

průměrná hodnota g na $(-1/2,\, 1/2)$

$$f * g(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy = \int$$

$$f(x-y) = \begin{cases} 0 & |x-y| \geq \frac{1}{2} \\ 1 & |x-y| < \frac{1}{2} \end{cases}$$

$$= \int_{x-1/2}^{x+1/2} g(y)\, dy$$

… průměrná hodnota g na $(x - 1/2,\, x + 1/2)$

PDR

Př: Rovnice vedení tepla $\quad u: \underset{x}{\mathbb{R}^3} \times \underset{\partial t}{(0,+\infty)} \to \mathbb{R}$ teplota

$$\partial_t u - \Delta u = 0 \quad \text{v} \quad \mathbb{R}^3 \times (0,+\infty) =: \Omega$$

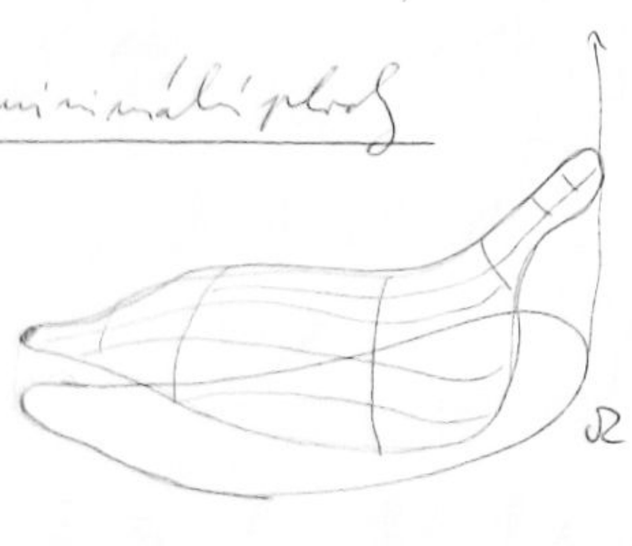
$$\partial_t u = \frac{\partial u}{\partial t}; \quad \Delta u = \sum_{i=1}^{3} \frac{\partial^2 u}{\partial x_i^2}$$ Laplaceův operátor

→ nestacionární, evoluční

stacionární varianta, $u: \mathbb{R}^3 \to \mathbb{R}$

$-\Delta u = 0$ Laplaceova rovnice

Rovnice minimální plochy



Symetrie: $a_{ij} + a_{ji} = b_{ij} \Rightarrow b_{ij} = b_{ji}$

$$a_{ij} \longrightarrow \frac{a_{ij} + a_{ji}}{2};$$

Pp: $A = (a_{ij})$ je symetrická matice

Zjednodušení hlavního členu

$$\sum_{i,j=1}^{d} a_{ij} \frac{\partial u}{\partial x_i \partial x_j} \,, \quad \text{uvažujeme } y = Px \,, \; P \in M^{d\times d}, x \in \mathbb{R}^d$$

$u(x) := v(Px)$, v je nová neznámá fce

$$\partial_{x_i} u(x) = \partial_{x_i} \Big[v\Big(\sum_j P_{1j} x_j, \dots, \sum_j P_{dj} x_j\Big)\Big]$$

$$= \sum_{k=1}^{d} \partial_{y_k} v(\dots) P_{ki}$$

$$\partial_{x_i}\partial_{x_j} u(x) = \sum_{k=1}^{d}\sum_{l=1}^{d} \partial_{y_l}\partial_{y_k} v(\dots) P_{ki} P_{lj}$$

$$\longrightarrow \sum_{k,l=1}^{d} \partial_{y_l}\partial_{y_k} v(\dots) \underbrace{\sum_{i,j=1}^{d} P_{ki}\, a_{ij}\, \underbrace{P_{lj}}_{(P^T)_{jl}}}_{(PAP^T)_{kl}}$$

$$(PAP^T)_{kl} = D_{kl} = \delta_{kl}\mu_k = \begin{cases} \mu_k & k = l \\ 0 & k \neq l \end{cases}$$

$$\longrightarrow \sum_{k=1}^{d} \partial^2_{y_k} v(\dots)\, \mu_k$$

$\longrightarrow$ Podobná transformace s $P = \mathrm{diag}\left(\frac{1}{\sqrt{|\mu_k|}}\right)_k \longrightarrow$

μ_k lze nahradit $\mathrm{sign}(\mu_k)$ $\quad \mu_k \neq 0$ jinak 0!