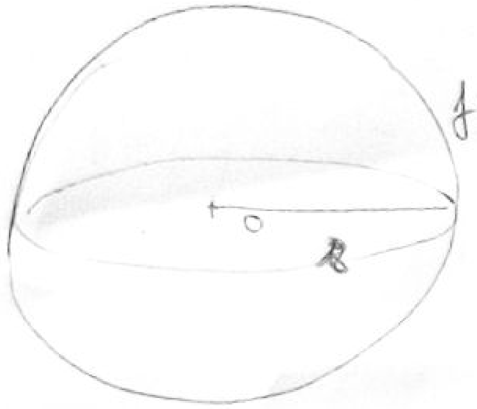
294. Fourierova transformace

Koule 4.2



$f(x) = R(r)\ ,\ r = |x|$

$$\hat{f}(\xi) = \int_{\mathbb{R}^3} f(x)\, e^{-2\pi i (x\cdot\xi)}\, dx \underset{(0,0,|\xi|)}{=} \int_{\mathbb{R}^3} f(x)\, e^{-2\pi i x_3 |\xi|}\, dx$$

$\xi \in \mathbb{R}^3$

$$= \left|\begin{array}{lll} \text{sférické souřadnice:} & & \\ x_1 = r\cos\alpha\cos\beta, & \alpha\in(-\frac{\pi}{2},\frac{\pi}{2}) & , J = r^2\cos\alpha \\ x_2 = r\cos\alpha\sin\beta & \beta\in(0,2\pi) & \\ x_3 = r\sin\alpha & r\in(0,+\infty) & \end{array}\right.$$

$$= \int_0^{+\infty} R(r) \int_0^{2\pi} \underbrace{\int_{-\frac{\pi}{2}}^{\pi/2} e^{-2\pi i r\sin\alpha |\xi|}\, r\cos\alpha\, d\alpha}_{(r\sin\alpha)' = r\cos\alpha,\ =: t}\, d\beta\, r\, dr =$$

$$= \int_0^{+\infty} r R(r)\, 2\pi \int_{-r}^{r} e^{-2\pi i t|\xi|}\, dt\, dr = \int_0^{+\infty} r R(r)\, 2\pi \left[\frac{e^{-2\pi i t|\xi|}}{-2\pi i|\xi|}\right]_{-r}^{r} dr$$

$$= \int_0^{+\infty} r R(r)\, \frac{1}{|\xi|} \left(\frac{e^{2\pi i r|\xi|} - e^{-2\pi i r|\xi|}}{2i}\right) dr = \int_0^{+\infty} r R(r)\, \frac{2}{|\xi|} \sin(2\pi r|\xi|)\, dr$$

Kveta 4.4 :

$$\widehat{f(x+z)}(\xi) = \hat{g}(\xi), \text{ kde } g(x) := f(x+z), \ z \text{ pevné}$$

$$\widehat{f(x+z)}(\xi) = \int_{\mathbb{R}^m} f(x+z)\, e^{-2\pi i (x,\xi)}\, dx = \left| \begin{matrix} y = x+z \\ J = 1 \end{matrix} \right|$$

$$= \int_{\mathbb{R}^m} f(y)\, e^{-2\pi i (y,\xi)} \cdot e^{2\pi i (z,\xi)}\, dy = e^{2\pi i (z,\xi)} \hat{f}(\xi)$$

$$(y-z,\xi) = (y,\xi) - (z,\xi)$$

$$\widehat{f(\alpha x)}(\xi) = \int_{\mathbb{R}^m} f(\alpha x)\, e^{-2\pi i (x,\xi)}\, dx = \left| \begin{matrix} y = \alpha x, \ \alpha \in \mathbb{R}, \ \alpha \neq 0 \\ x = \frac{y}{\alpha} \quad x, y \in \mathbb{R}^m \\ = \varphi(y) \end{matrix} \right|$$

$$\nabla \varphi(y) = \begin{pmatrix} \frac{1}{\alpha} & 0 & \cdots & 0 \\ 0 & \frac{1}{\alpha} & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & \cdots \ 0 & \frac{1}{\alpha} \end{pmatrix} \qquad \varphi(y) = \begin{pmatrix} \frac{y_1}{\alpha} \\ \vdots \\ \frac{y_m}{\alpha} \end{pmatrix}$$

$$\operatorname{Jac} \varphi(y) = |\det \nabla \varphi| = \left| \left(\frac{1}{\alpha}\right)^m \right|$$

$$\widehat{f(\alpha x)}(y) = \int_{\mathbb{R}^m} f(y)\, e^{-2\pi i \left(\frac{y}{\alpha}, \xi\right)} \left|\frac{1}{\alpha}\right|^m dy = \left|\frac{1}{\alpha}\right|^m \hat{f}\left(\frac{\xi}{\alpha}\right)$$

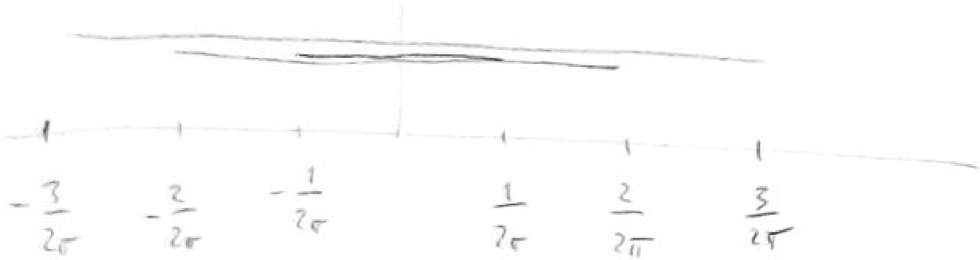
Př:
$$\widehat{\chi_{(-1,1)}}(\xi) = \frac{\sin 2\pi \xi}{\pi \xi}$$

$$\widehat{\chi_{(-\alpha,\alpha)}}(\xi) \overset{\text{Věta 4.3}}{=} \alpha\, \widehat{\chi_{(-1,1)}}(\alpha \xi) = \widehat{\chi_{(-1,1)}\left(\frac{x}{\alpha}\right)}(\xi)$$

$\alpha > 0$
$$\chi_{(-\alpha,\alpha)}(x) = \chi_{(-1,1)}\left(\frac{x}{\alpha}\right)$$

$$\widehat{\chi_{(-\alpha,\alpha)}}(\xi) = \alpha\, \frac{\sin 2\pi\alpha\xi}{\pi\alpha\xi} = \frac{\sin 2\pi\alpha\xi}{\pi\xi}$$

$$\alpha := \frac{n}{2\pi} : \quad \widehat{\chi_{\left(-\frac{n}{2\pi},\, \frac{n}{2\pi}\right)}}(\xi) = \frac{\sin n\xi}{\pi\xi}$$



Cvičení

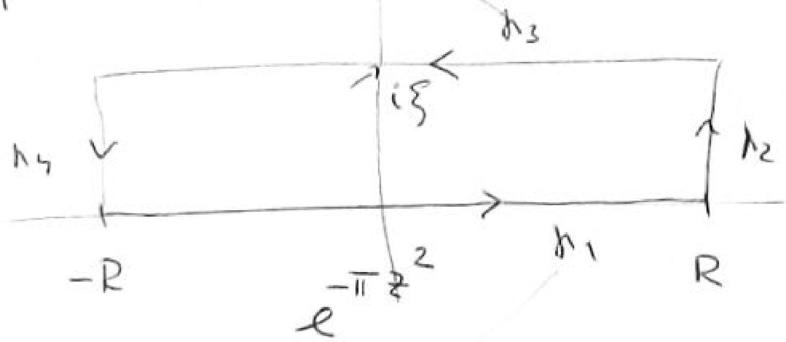
$$\widehat{e^{-\pi x^2}}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi}\,dx =$$

$$= \underbrace{\int_{\mathbb{R}} e^{-\pi\left(x^2+2ix\xi+(i\xi)^2\right)}\,dx}_{\overset{?}{=}1}\; e^{-\pi\xi^2}$$

R → +∞ (−)

$$\int_{\mathbb{R}} e^{-\pi(x+i\xi)^2}\,dx \overset{?}{=} 1$$

|| reziduová věta



$$\int_{\mathbb{R}} e^{-\pi x^2}\,dx$$

R → +∞

$$\int_{\gamma_2} + \int_{\gamma_4} \xrightarrow{R\to+\infty} 0$$

||

$$\sqrt{\int_{\mathbb{R}} e^{-\pi x^2}\,dx \int_{\mathbb{R}} e^{-\pi y^2}\,dy} = \sqrt{\int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\,d(x,y)} = \left|\begin{array}{l} x = r\cos\alpha \\ y = r\sin\alpha \\ \mathrm{Jac} = r \end{array}\right.$$

$$= \int_0^{+\infty}\int_0^{2\pi} e^{-\pi r^2}\, r\,d\alpha\,dr = 2\pi\left[\frac{e^{-\pi r^2}}{-2\pi}\right]_0^{+\infty} = 1$$

K důk věty 5.4:

Jak ho nedělat! Dokážeme inverzní fci:

$$\check{\hat{f}}(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} f(\xi)\, e^{-2\pi i (\xi, y)}\, d\xi\, e^{+2\pi i (y \cdot x)}\, dy$$

$$= \underbrace{\int_{\mathbb{R}^n} \underbrace{\int_{\mathbb{R}^n} f(\xi)\, e^{-2\pi i (\xi, y)}\, d\xi}_{\hat{f}(y)} \cdot e^{2\pi i (y, x)}\, dy}_{\check{\hat{f}}(x)}$$

$$\overset{?}{\cancel{=}} \int_{\mathbb{R}^n \times \mathbb{R}^n} f(\xi)\, \underbrace{e^{2\pi i (y,x) - (\xi, y)}}_{e^{2\pi i (y, x - \xi)}}\, d(\xi, y)$$

obecně není integrovatelná fce

Rychle klesající fce

1) $D^\alpha \varphi = \dfrac{\partial^{\alpha_1+\dots+\alpha_m}}{\partial^{\alpha_1} x_1 \, \partial^{\alpha_2} x_2 \dots \partial^{\alpha_m} x_m}$

$\alpha = (\alpha_1, \dots, \alpha_m) \in (\mathbb{N}_0)^m$

2) Všechny derivace jsou omezené ($q = 0$).

3) Libovolná derivace klesá $\underbrace{\text{rychleji}}_{\text{v relaci}}$ než $\dfrac{1}{\text{libovolný polynom}}$.

Motivace pro zavedení Schwartzova prostoru

<u>$m = 1$</u>:

$$\widehat{f'}(\xi) = \int_{\mathbb{R}} \underset{\downarrow i}{f'(x)} \; \underset{\downarrow d}{e^{-2\pi i x \xi}} \, dx =$$

$$\qquad\qquad f(x) \qquad -2\pi i \xi$$

$$= +2\pi i \xi \int_{\mathbb{R}} f(x)\, e^{-2\pi i x \xi} \, dx = 2\pi i \xi \, \hat{f}(\xi)$$

$\uparrow$

smizely hraniční členy – pp.

Pzn: $\mathcal{S}(\mathbb{R}^m) \subset L^1(\mathbb{R}^m)$

$$f \in \mathcal{S}(\mathbb{R}^m) \Rightarrow \exists K > 0 \ \forall x \in \mathbb{R}^m : (1 + |x|^2)^m |f(x)| < K$$

$$\Rightarrow |f(x)| \le \frac{K}{(1+|x|^2)^m} \in L^1(\mathbb{R}^m) \Rightarrow f \in L^1(\mathbb{R}^m)$$

Důk V 5.5 : $n = 1$

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)\, e^{-2\pi i (x,\xi)}\, dx$$

$\hat{f}$ je spojitá podle věty o spojité závislosti int. na parametru.

- měřitelnost v x ✓
- spojitost v ξ ✓
- integrabilní majoranta $\left| f(x)\, e^{-2\pi i (x,\xi)} \right| \le |f(x)|$, $f \in L^1(\mathbb{R})$

$\Rightarrow$ $\hat{f}$ je spojitá na $\mathbb{R}$

$\Rightarrow$ $\forall \xi \in \mathbb{R} : |\hat{f}(\xi)| \le \|f\|_1$

$\longrightarrow$ $|\alpha| = \sum_{i=1}^{n} \alpha_i$

$$Y = \hat{y}$$

$$\widehat{y'' - y} = \widehat{e^{-\pi x^2}}$$

$$\widehat{y''} - \hat{y} = e^{-\pi \xi^2}$$

$$(2\pi i \xi)^2 \hat{y}(\xi) - \hat{y}(\xi) = e^{-\pi \xi^2}$$

$$-\hat{y}(\xi)\left(4\pi^2 \xi^2 + 1\right) = e^{-\pi \xi^2}$$

$$\hat{y}(\xi) = -\frac{e^{-\pi \xi^2}}{1 + 4\pi^2 \xi^2} = -\hat{g}(\xi) \cdot \widehat{e^{-\pi x^2}}(\xi)$$

kde $\hat{g}(\xi) = \dfrac{1}{1 + 4\pi^2 \xi^2}$ $\longrightarrow$ je potřeba spočítat g

$$\hat{y}(\xi) = -\widehat{g * e^{-\pi x^2}}(\xi)$$

$$y(\cancel{\xi}x) = -\left(g * e^{-\pi x^2}\right)(x)$$