Algorithms and datastructures I
Lecture 10: universal hashing and divide & conquer

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Hash functions

Hash function is a function h from universe $\mathcal{U}$ to set $\mathcal{P} = \{0, 1, \ldots, p-1\}$ (of hashes).

Hash table with separate chaining for set $S \subseteq U$ with hash function $h \colon \mathcal{U} \to \mathcal{P}$.

Hash table is an array H of linked lists indexed by $\mathcal{P}$. List $H[i]$ contains all elements e of set S such that $h(e) = i$.

Assumptions

1. $h(x)$ can be computed in $O(1)$.
2. $h(x)$ "behaves randomly".

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Observation

Every entry of the hash table will contain approximately $\frac{|S|}{p}$ elements.

Corollary

Operations FIND, INSERT and DELETE will run in $O(|S|)$ however expected (average) runtime is only $O(\frac{|S|}{p})$.

Corollary

Putting $p \sim |S|$ we get FIND, INSERT and DELETE is running on average approximately in $O(1)$.

Universal hashing

Definition (c-universal system of hash functions)

System $\mathcal{S}$ of hash functions from universe $\mathcal{U}$ to $\{0, 1, \ldots, p-1\}$ is c-universal for given $c \geq 1$ if for every $x, y \in \mathcal{U}$, $x \neq y$

$$\Pr_{h \in \mathcal{S}}[h(x) = h(y)] \leq \frac{c}{p}.$$

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Lemma

Let $\mathcal{S}$ be c-universal system of hash functions $\mathcal{U} \to \{0, 1, \ldots, p\}$. Let $x_1, x_2, \ldots, x_n, y$ be pairwise different elements of $\mathcal{U}$. Then

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The lemma shows expected runtime of INSERT, FIND and DELETE with separate chaining.

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Proof.

We define indicators $I_1, I_2, \ldots, I_n$:

$$I_i = \begin{cases} 0 & \text{if } h(x_i) \neq h(y) \\ 1 & \text{if } h(x_i) = h(y). \end{cases}$$

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□

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System of functions $\mathcal{S}\colon \mathbb{Z}_p^d \to \{0, 1, \ldots, p-1\}$

Let p be a prime number, $\mathcal{P} = \mathbb{Z}_p$ (ring modulo p), $\mathcal{U} = \mathbb{Z}_p^d$ (vectors of length d in $\mathbb{Z}_p$).

$$\mathcal{S} = \{h_{\vec{a}}\colon \vec{a} \in \mathbb{Z}_p^d, \vec{a} \neq 0\} \text{ where } h_{\vec{a}}(x) = \vec{a}\vec{x} = \sum_{i=1}^{d} a_i x_i \mod p. \ (a_i x_i \text{ is the scalar product}).$$

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Set $\vec{x} \neq \vec{y} \in \mathbb{Z}_p^d$. WLOG $x_d \neq y_d$. What is $\Pr_{\vec{a} \in \mathbb{Z}_p^d}[\vec{a}\vec{x} = \vec{a}\vec{y} \mod p]$?

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Put $\vec{z} = \vec{x} - \vec{y}$. ($\vec{a}\vec{x} \equiv \vec{a}\vec{y}$ means $\vec{a}\vec{x} = \vec{a}\vec{y} \mod p$)

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$\sum_{i=1}^{d-1} a_i z_i + a_d z_d \equiv 0$ happens only if $\sum_{i=1}^{d-1} a_i z_i \equiv -a_d z_d$. This has probability $\frac{1}{p}$. □

Divide & Conquer

"Divide and conquer is an algorithm design paradigm based on multi-branched recursion. A divide-and-conquer algorithm works by recursively breaking down a problem into two or more sub-problems of the same or related type, until these become simple enough to be solved directly. The solutions to the sub-problems are then combined to give a solution to the original problem."

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MergeSort $(x_1, \ldots, x_n)$, John von Neumann, 1945

1. $(y_1, \ldots, y_{\lfloor \frac{n}{2} \rfloor}) \leftarrow$ MergeSort $(x_1, \ldots, x_{\lfloor \frac{n}{2} \rfloor})$
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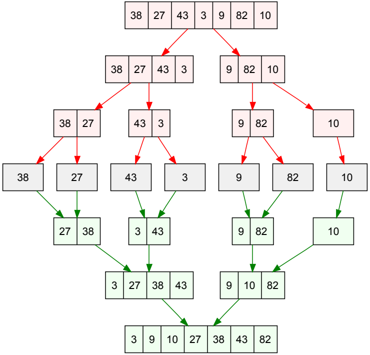

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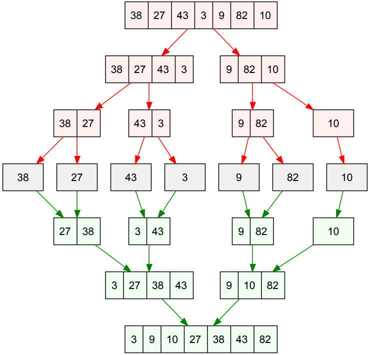
Time complexity (for $n = 2^k$)

$$T(n) = 2T(\frac{n}{2}) + \Theta(n)$$

$$T(1) = 1$$



Solving the recurrence: method 1 (by substitution)

Time complexity (for $n = 2^k$)

$$T(n) = 2T(\frac{n}{2}) + cn$$

$$T(1) = 1$$

$$T(n) = 2\left(2T(\frac{n}{4}) + \frac{cn}{2}\right) + cn$$

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&= n + cn \log n = \Theta(n \log n)
\end{aligned}$$

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Memory complexity (for $n = 2^k$)

$$R(n) = R(\frac{n}{2}) + \Theta(n)$$

$$R(1) = 1$$

Solving the recurrence: method 1 (by substitution)

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$$R(n) = \Theta(n) + \Theta(\frac{n}{2}) + \cdots + \Theta(\frac{n}{2^k}) = \Theta(n)$$

Solving the recurrence: method 2 (tree of recursion)

Time complexity (for $n = 2^k$)

$$T(n) = 2T(\frac{n}{2}) + cn$$

$$T(1) = 1$$

Tree of recursion:

# of subprob	size of subprob.	time per subprob	time per level

Multiplication (Karatsuba 1960)

$$X = \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B$$

$$Y = \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D$$

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$$X \cdot Y = AC \cdot 10^{n} + (AD + BC) \cdot 10^{\frac{n}{2}} + BD$$

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$$T(n) = 4T(\frac{n}{2}) + cn$$

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X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\
X \cdot Y &= AC \cdot 10^{n} + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\
&= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
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\end{aligned}
$$

$$T(n) = 3T(\frac{n}{2}) + cn$$

Anatolii Alexeievitch Karatsuba

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
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&= AC \cdot 10^{n} + ((A + B)(C + D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

$$T(n) = \color{red}{3}\color{black}{}T(\frac{n}{2}) + cn$$

# of subprob	size of subprob.	time per subprob	time per level

Multiplication (Karatsuba 1960)

$$X = \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B$$

$$Y = \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D$$

$$X \cdot Y = AC \cdot 10^n + (AD + BC) \cdot 10^{\frac{n}{2}} + BD$$

$$= AC \cdot 10^n + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$T(n) = \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^i cn$$

Multiplication (Karatsuba 1960)

$$X = \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B$$

$$Y = \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D$$

$$X \cdot Y = AC \cdot 10^n + (AD + BC) \cdot 10^{\frac{n}{2}} + BD$$

$$= AC \cdot 10^n + ((A + B)(C + D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$T(n) = \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^i cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n+1} - 1}{\left(\frac{3}{2}\right) - 1}$$

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\
X \cdot Y &= AC \cdot 10^{n} + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\
&= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

$$T(n) = \color{red}{3}\color{black}{}T\left(\frac{n}{2}\right) + cn$$

$$T(n) = \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n + 1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right)$$

Multiplication (Karatsuba 1960)

$$\begin{aligned} X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\ Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\ X \cdot Y &= AC \cdot 10^{n} + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\ &= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD \end{aligned}$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$T(n) = \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n+1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right)$$

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
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\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n + 1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\
&= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right)
\end{aligned}
$$

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
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&= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n+1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\
&= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right) = \Theta\left(2^{\log 3 \log n}\right)
\end{aligned}
$$

Multiplication (Karatsuba 1960)

$$\begin{aligned} X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\ Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\ X \cdot Y &= AC \cdot 10^n + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\ &= AC \cdot 10^n + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD \end{aligned}$$

Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^i cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n + 1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\ &= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right) = \Theta\left(2^{\log 3 \log n}\right) = \Theta\left(\left(2^{\log n}\right)^{\log 3}\right) \end{aligned}$$

Multiplication (Karatsuba 1960)

$$\begin{aligned} X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\ Y &= \boxed{C}\boxed{D} = C \cdot 10^{\frac{n}{2}} + D \\ X \cdot Y &= AC \cdot 10^n + (AD + BC) \cdot 10^{\frac{n}{2}} + BD \\ &= AC \cdot 10^n + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD \end{aligned}$$

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$$T(n) = 3T(\frac{n}{2}) + cn$$

$$\begin{aligned} T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^i cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n + 1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\ &= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right) = \Theta\left(2^{\log 3 \log n}\right) = \Theta\left(\left(2^{\log n}\right)^{\log 3}\right) = \Theta\left(n^{\log 3}\right) \end{aligned}$$

Multiplication (Karatsuba 1960)

$$
\begin{aligned}
X &= \boxed{A}\boxed{B} = A \cdot 10^{\frac{n}{2}} + B \\
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&= AC \cdot 10^{n} + ((A+B)(C+D) - AC - BD) \cdot 10^{\frac{n}{2}} + BD
\end{aligned}
$$


Anatolii Alexeievitch Karatsuba

$$T(n) = 3T(\frac{n}{2}) + cn$$

$$
\begin{aligned}
T(n) &= \sum_{i=0}^{\log n} \left(\frac{3}{2}\right)^{i} cn = cn \cdot \frac{\left(\frac{3}{2}\right)^{\log n+1} - 1}{\left(\frac{3}{2}\right) - 1} = \Theta\left(n \cdot \left(\frac{3}{2}\right)^{\log n}\right) = \Theta\left(n \cdot \left(\frac{3^{\log n}}{2^{\log n}}\right)\right) = \Theta(3^{\log n}) \\
&= \Theta\left(\left(2^{\log 3}\right)^{\log n}\right) = \Theta\left(2^{\log 3 \log n}\right) = \Theta\left(\left(2^{\log n}\right)^{\log 3}\right) = \Theta\left(n^{\log 3}\right) = \Theta\left(n^{1.59\cdots}\right).
\end{aligned}
$$