# Algorithms and datastructures I
# Lecture 9: RB-trees and hashing

Jan Hubička

Department of Applied Mathematics
Charles University
Prague

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# Set datastructure

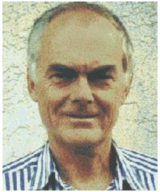
We would like to represent a set (or a dictionary) of some elements from an universe.
We expect that elements of the universum in set can be assigned and compared in $O(1)$.

- INSERT($v$): Insert $v$ to the set.
- DELETE($v$): Delete $v$ from the set.
- FIND($v$): Find $v$ in the set.
- MIN: Return minimum.
- MAX: Return maximum.
- SUCC($v$): Find successor.
- PRED($v$): Find predecessor.

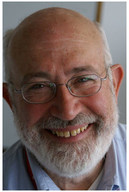
## Basic implementations

| | INSERT | DELETE | FIND | MIN/MAX | SUCC/PRED |
|---|---|---|---|---|---|
| Linked list | $O(n)$ or $O(1)$ | $O(n)$ or $O(1)$ | $O(n)$ | $O(n)$ | $O(n)$ |
| Array | $O(n)$ or $O(1)$ | $O(n)$ or $O(1)$ | $O(n)$ | $O(n)$ | $O(n)$ |
| Sorted array | $O(n)$ | $O(n)$ | $O(\log n)$ | $O(1)$ | $O(\log n)$ or $O(1)$ |
| binary search trees | $O(n)$ | $O(n)$ | $O(n)$ | $O(n)$ | $O(n)$ |
| AVL-trees | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |
| $(r, b)$-trees | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ |

# $(a, b)$-trees (Bayer, McCreight)



Rudolf Bayer



Edward M. McCreight

**Definition (Generalized search tree)**

Generalised search tree is a rooted tree with specified order of sons and two types of vertices:

1. Internal vertices contains non-zero number of keys. If internal vertex has keys $x_1 < \cdots < x_k$ then it has $k+1$ sons $s_0, \ldots, s_k$. Keys separate values in sons, so:
   $T(s_0) < x_1 < T(s_1) < x_2 < \cdots < x_{k-1} < T(s_{k-1}) < x_k < T(s_k)$
2. External vertices contain no keys and are leaf.

**Definition ($(a, b)$-tree)**

$(a, b)$-tree for a given $a \geq 2$, $b \geq 2a - 1$ is a generalised search tree such that:

1. Root has 2 to $b$ sons.
2. Other internal vertices have $a$ to $b$ sons.
3. All external vertices are in the level.

**Lemma**

*Every $(a, b)$-tree with $n$ keys has depth $\Theta(\log n)$.*

# Insert to $(a, b)$-tree

**Insert($v$,$x$)**

Let $u$ be the last internal vertex visited by Find($v$,$x$).

1. If $u$ contains $x$ return.
2. Otherwise add $x$ into $u$ and insert new external vertex
3. If $u$ has more than $b$ sons, split it possibly recursing to father.

It is possible to split preventively if $b \geq 2a$. We will use it today.

# Red-black trees (Bayer 1972; Guibas, Sedgewick 1978; Anderson 1993; Sedgewick 2008)

We can represent $(2, 4)$-trees using binary search trees with colored edges.

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**Definition (Left leaning red-back tree)**

LLRB-tree is binary search tree with external vertices and edges colored either red or black. It satisfies:

1. There are no two red edges adjacent to each other.
2. If there is only one red edge from a vertex then it is left.
3. Edges to leaves are always black.
4. Every path from root to leaf goes through the same number of black edges.

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Optimization: Color of edge may be stored in its destination vertex.

# Depth of LLRB-trees

**Lemma**

*Every LLRB-tree with $n$ keys has depth $\Theta(\log n)$.*

**Proof.**

We know that every LLRB-tree tree corresponds to an $(2, 4)$-tree of height $h = \Theta(\log n)$.
The height $h'$ of LLRB tree is $h \leq h' \leq 2h$. □

# Operations on LLRB-trees

**Observation**

Operations FIND, MIN, MAX, SUCC and PRED run in $\Theta(\log n)$.

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Operations INSERT and DELETE can be derived from ones on $(2, 4)$-trees.

# INSERT to an LLRB-tree

Lets see how insertion to $(2, 4)$-tree with preventive splitting translates to RB-tree.

**Insert($v$,$x$)**

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Exchanging steps 3 and 7 leads to representation of $(2, 3)$-trees.

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Exchanging steps 3 and 7 leads to representation of $(2, 3)$-trees.
Fact: DELETE can also be implemented in $\Theta(\log n)$ time.

**Theorem**

*Operations INSERT, DELETE, FIND, MIN, MAX, SUCC and PRED on LLRB-tree runs in $\Theta(\log n)$ time.*


And Now For Something
Completely Different

# Tries

Let $\Sigma$ be a fixed alphabet. Let $S \subseteq \Sigma^*$ be a set of words over alphabet $\Sigma$.

**Definition (Trie: middle of retrieval, invented by René de la Briandais in 1959; named by Edward Frenklin)**

Trie for some set of words $S$ is a rooted tree where

1. vertices are all prefixes of words $W \in X$, and
2. $W'$ is a son of word $W$ if $W'$ is created from $W$ by extending it by one letter.

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To store sets of integers one can see integers as words in some fixed base. Result is known as a radix tree.

# Amortised complexity

Insertion to a (dynamically allocated) growing array.

**Insert($(A, s, n)$,$x$) insert element $x$ to array $A$ of size $s$ containing $n$ elements**

1. if $n = s$:
2. Allocate array $A'$ of size $2s$.
3. For $i = 0, 1, \dots, n-1$: $A'[i] \leftarrow A[i]$.
4. Free $A$.
5. $A \leftarrow A'$, $s \leftarrow 2s$
6. $A[n] \leftarrow x$, $n \leftarrow n + 1$
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Worst case complexity of INSERT is $O(n)$.

**Theorem**

*Performing $n$ operations* INSERT *starting from the empty array will run in time $\Theta(n)$.*

**Proof.**

To insert $2^i$ elements one needs $2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^{i-1} = 2^i - 1$ copy operations. □

# Hash functions

Hash function is a function $h$ from universe $\mathcal{U}$ to set $\mathcal{P} = \{0, 1, \ldots, p-1\}$ (of hashes).

**Hash table with separate chaining for set $S \subseteq U$ with hash function $h\colon \mathcal{U} \to \mathcal{P}$.**

Hash table is an array $H$ of linked lists indexed by $\mathcal{P}$. List $H[i]$ contains all elements $e$ of set $S$ such that $h(e) = i$.

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**Corollary**

Putting $p \sim |S|$ we get Find, Insert and Delete is running on average approximately in $O(1)$.

# Hash functions

**Example (Integers: $h\colon \mathbb{N} \to \{0, 1, \ldots, p-1\}$)**

$$h(x) = ax \mod p$$

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$$h(x) = \left( \sum_{i=1}^{|x|} x_i a^{|x|-i} \mod p \right).$$

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Can be effectively computed as (Horner's method):

$$\begin{aligned}
h_1 &= x_1 \\
h_2 &= (h_1 a + x_2) \mod p \\
h_3 &= (h_2 a + x_3) \mod p \\
\dots & \quad \dots \\
h_{|x|} &= (h_{|x|-1} a + x_{|x|}) \mod p
\end{aligned}$$

# Open addressing

An alternative way of solving collisions is to use hash function $h(x, i)$ such that for every $x \in \mathcal{U}$ sequence $h(x, 0), h(x, 1), \ldots, h(x, p-1)$ is a permutation of $(0, 1, \ldots, p-1)$.

**Insert(x)**

1. For $i = 0, \ldots, p-1$:
2. $\quad j \leftarrow h(x, i)$
3. $\quad$ If $H[j] = \emptyset$: put $H[j] \leftarrow x$ and return.
4. Report that table is full.

**Find(x)**

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We can not remove values from the table, just mark them as removed.

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