Izolované singularity

Př: • $f(z) = \begin{cases} z & z \neq 0 \\ \text{nedef.} & z = 0 \end{cases}$

$f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$

→ f má v 0 izolovanou singularitu → odstranitelnou singularitu

$$\lim_{z \to 0} f(z) = 0 \in \mathbb{C}$$

→ f má v 1 izolovanou sing. – odstranitelnou sing.

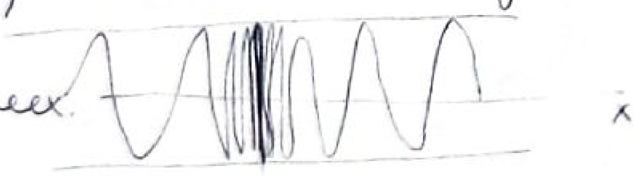
• $f(z) := \begin{cases} \frac{1}{z} & \text{v } z \neq 0 \\ \text{nedef.} & \text{v } z = 0 \end{cases}$ $\qquad f \in \mathcal{H}(\mathbb{C} \setminus \{0\})$

⇒ 0 je izolovaná singularita f

$\lim_{z \to 0} f(z) = \infty$ ⇒ 0 je pól

• $f(z) := \sin\left(\frac{1}{z}\right)$ má izolovanou sing. v 0

$\lim_{z \to 0} \sin\left(\frac{1}{z}\right)$ neex.



f má v 0 podstatnou singularitu

K Větě 3.11: 1 => 2 ~~=> 3~~

2 => 3

$$f(z) = \sum_{n=-\infty}^{+\infty} a_n (z-a)^n \quad \text{na } P(a)$$

$$a_n = \frac{1}{2\pi i} \int_{\gamma_{a,r}} \frac{f(w)}{(w-a)^{n+1}}\, dw$$

} V 3.10

Ukážeme, že pro $n \in \mathbb{Z},\ n \le -1 : a_n = 0$

Víme $\exists K > 0 : |f(w)| < K$ na $P(a) \ni w$

$$|a_n| = \left| \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(\gamma_{a,r})}{r^{n+1} e^{i(n+1)t}} \cdot r i e^{it}\, dt \right| \qquad \gamma_{a,r}(t) = a + r e^{it},\ t \in [0, 2\pi]$$

$$\le \frac{1}{2\pi} \int_0^{2\pi} \frac{|f(\gamma_{a,r}(t))|}{r^{n+1}} \cdot r\, dt \le \frac{K}{2\pi} \int_0^{2\pi} r^{-n}\, dt = r^{-n} K \xrightarrow[r \to 0+]{} 0$$

$-n > 0$

$\Rightarrow a_n = 0$

$$\Rightarrow f(z) = \sum_{n=0}^{+\infty} a_n (z-a)^n \quad \text{na } P(a)$$

3 => 4 bylo   4 => 1

Př: v $\mathbb{R}$ $\quad f(x) = |x|$ $\quad x \neq 0$, nedef. $x = 0$ $\qquad \lim_{x \to 0} f(x) = 0$ ex.

ale $f'(0)$ neex. ani po dodef. lim

KV3.12

$1 \Rightarrow 2$ $\quad g(z) := \frac{1}{f(z)}$ na $P(a)$, kde $f \neq 0$

$$\lim_{z \to a} g(z) = 0 \text{ existuje konečná}, \; g \in \mathcal{H}(P(a))$$

$\Rightarrow$ g má v a odstranitelnou sing.

$$\Rightarrow g(z) = \sum_{n=n_0}^{+\infty} a_n (z-a)^n \; ; \quad a_{n_0} \neq 0$$

$$= (z-a)^{n_0} \underbrace{\sum_{n=n_0}^{+\infty} a_n (z-a)^{n-n_0}}_{h(z)}$$

$h \in \mathcal{H}(P(a))$
$\mathcal{H}(a)$
$h(a) = a_{n_0} \neq 0$

$$\frac{1}{f(z)} = (z-a)^{n_0} h(z)$$

$$f(z) = \frac{1}{(z-a)^{n_0}} \frac{1}{h(z)}$$

$$\lim_{z \to a} f(z) \cdot (z-a)^{n_0} = \lim_{z \to a} \frac{1}{h(z)} = \frac{1}{h(a)} \in \mathbb{C}$$

$$\lim_{z \to a} f(z) (z-a)^{n_0 - 1} = \lim_{z \to a} \frac{1}{h(z)} \cdot \frac{1}{z-a} = \infty$$

$2 \Rightarrow 3$ $\quad g(z) := (z-a)^{n_0} f(z)$ má odstranitelnou sing. v a

bez dalších detailů

$3 \Rightarrow 1$ triv.

K⇔ V 3.13:

1 ⇒ 2   ať 2 neplatí $\exists\, w \in \mathbb{C} \cup \{\infty\}$ a $\varepsilon > 0$

$$f(P(a)) \cap U(w, \varepsilon) = \emptyset$$

pp. $w \in \mathbb{C}$

$$g(z) = \frac{1}{f(z) - w} \quad \dots \quad g \in \mathcal{H}(P(a))$$

$$z \in P(a) : |f(z) - w| \geq \varepsilon$$

$$|g(z)| \leq \frac{1}{|f(z) - w|} \leq \frac{1}{\varepsilon}$$ je omezená na $P(a)$

⇒ g má v a odstranitelnou sing.

$$\frac{1}{f(z) - w} = g(z) = \sum_{n=n_0}^{+\infty} a_n (z-a)^n \quad , a_{n_0} \neq 0$$

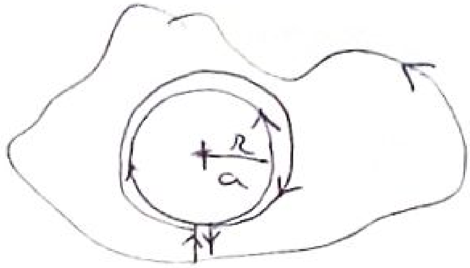
$$= (z-a)^{n_0} h(z) \qquad h \in \mathcal{H}(U(a)),\ h(a) \neq 0,\ = a_{n_0}$$

$$f(z) = w + \frac{1}{(z-a)^{n_0}} \cdot \underbrace{\frac{1}{h(z)}}_{\substack{\in \mathcal{H}(U(a)) \\ \neq 0}}$$

$\overset{V3.12}{\Rightarrow}$ f má v a pól ↯ s(1)

2 ⇒ 3 bez dk
3 ⇒ 1

Residuum



$\gamma(t) = a + r e^{it},\ t \in [0, 2\pi]$

$$\int_\gamma f(z)\,dz \overset{!?}{=} \sum_{n=-\infty}^{+\infty} a_n \underbrace{\int_\gamma (z-a)^n\,dz}_{=\begin{cases} 0 & n \neq -1 \\ 2\pi i & n=-1 \end{cases}} = a_{-1}\cdot 2\pi i = 2\pi i \operatorname{res}_a f$$

K Větě 3.12

1) plyne z V3.11 $-\ f(z) = \sum_{n=0}^{+\infty} a_n (z-a)^n$

2) $f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + \cdots$

$g(z) = \qquad b_0 + b_1(z-a) + \cdots$

$f(z)g(z) = \frac{a_{-1}b_0}{z-a} + (a_0 b_0 + a_{-1} b_1) + (z-a)(a_{-1}b_2 + a_0 b_1 + a_1 b_0) \cdots$

$\operatorname{res}_a(fg) = a_{-1} b_0 = \operatorname{res}_a f \cdot g(a)$

3) Stačí ukázat: $\operatorname{res}_a \frac{1}{g} = \frac{1}{g'(a)}$ a $\frac{1}{g}$ má v a pól nás. 1

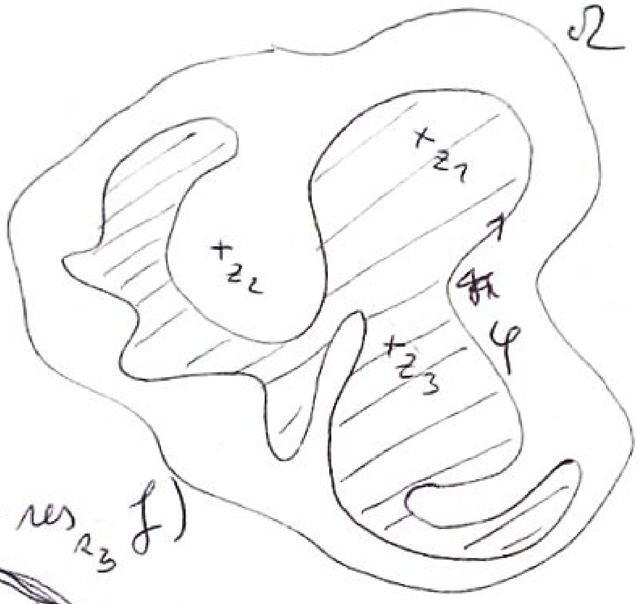
$g(z) = \overset{g'(a)}{b_1}(z-a) + b_2(z-a)^2 + \cdots$

$= (z-a)\underbrace{(b_1 + b_2(z-a) + \cdots)}_{h(z),\ h \in \mathcal{H}(U(a)),\ h(a) = b_1 \neq 0}$

$\frac{1}{g(z)} = \frac{1}{z-a}\cdot\frac{1}{h(z)}$, $\operatorname{res}_a \frac{1}{g} = \operatorname{res}_a \frac{1}{z-a}\cdot\frac{1}{h(z)} \overset{2)}{=} \frac{1}{h(a)} = \frac{1}{b_1} = \frac{1}{g'(a)}$

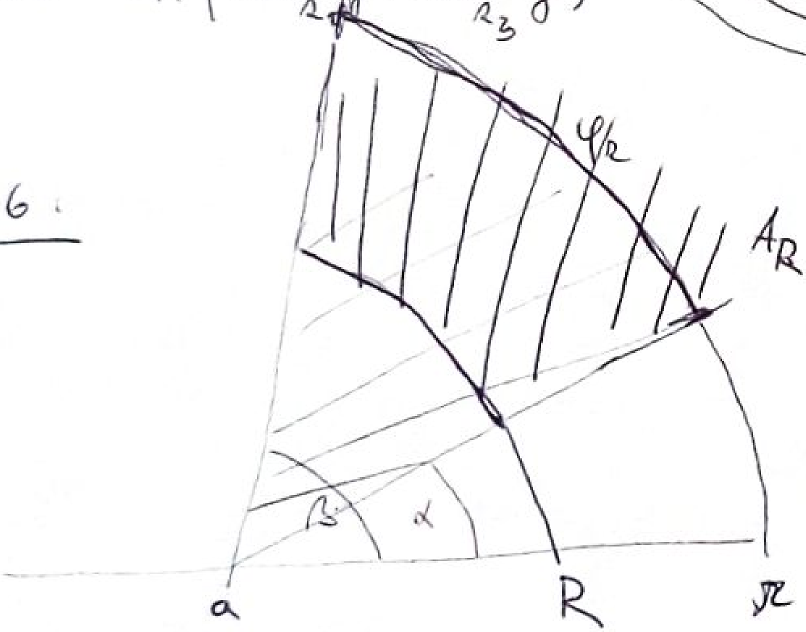
4) bez dk

K reziduové větě:

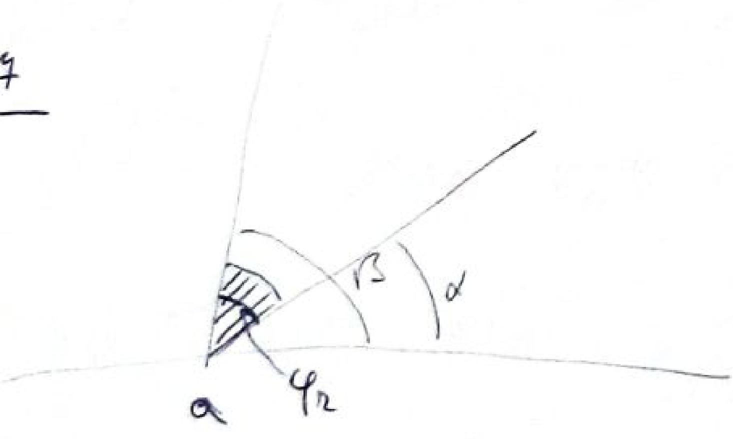


$$\int_\varphi f(z)\,dz = 2\pi i\left(\mathrm{res}_{z_1} f + \mathrm{res}_{z_3} f\right)$$

L 3.16:



L 3.17

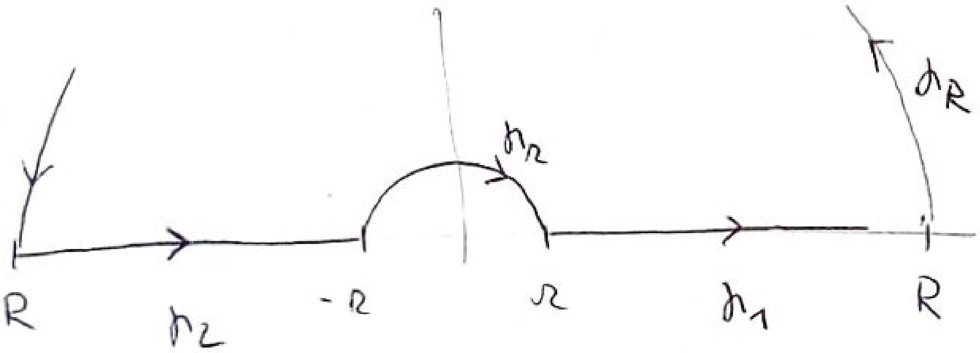


$$(\gamma)\int_0^{+\infty} \frac{\sin x}{x}\,dx = I$$

$$f(z) = \frac{e^{iz}}{z}, \quad z \in \mathbb{C}\setminus\{0\}; \quad f \in \mathcal{H}(\mathbb{C}\setminus\{0\})$$

$$f(x) = \frac{e^{ix}}{x} = \frac{\cos x}{x} + i\,\frac{\sin x}{x}, \quad x \in \mathbb{R}$$

γ :



γ_1, γ_2 souvisí s I

$\gamma_r \to -\pi i \operatorname{res}_0 f$ L 3.17 $r \to 0+$

$\gamma_R \to 0$ L 3.16 $R \to +\infty$

$\Rightarrow I = ?$ $\gamma_1(t) = t,\ t \in [r, R],\ \gamma_2(t) = t,\ t \in [-R, -r]$

$$\int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_r^R \frac{e^{ix}}{x}\,dx + \int_{-R}^{-r} \frac{e^{ix}}{x}\,dx =$$

$$\int_r^R \frac{e^{ix}}{x}\,dx + \int_r^R \frac{e^{-ix}}{-x}\,dx = \int_r^R \frac{1}{x}\left(e^{ix} - e^{-ix}\right)dx =$$

$$\int_r^R 2i\,\frac{\sin x}{x}\,dx. \quad \text{Cauchyova věta: } \left(\int_{\gamma_1} + \int_{\gamma_2} + \int_{\gamma_r} + \int_{\gamma_R}\right) f(z)\,dz = 0$$

$$\lim_{r \to 0+},\ \lim_{R \to +\infty}:\ 2i\int_0^{+\infty} \frac{\sin x}{x}\,dx = \pi i \operatorname{res}_0 f = \pi i$$

$$\Rightarrow \boxed{\int_0^{+\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}}$$