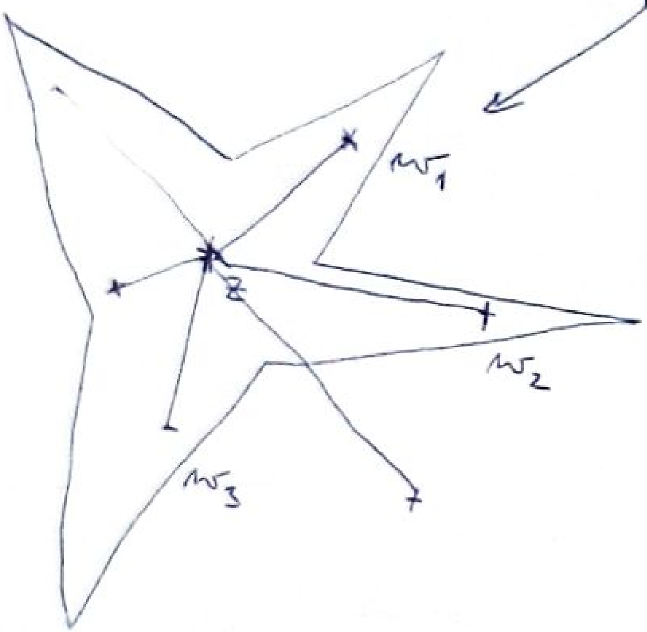
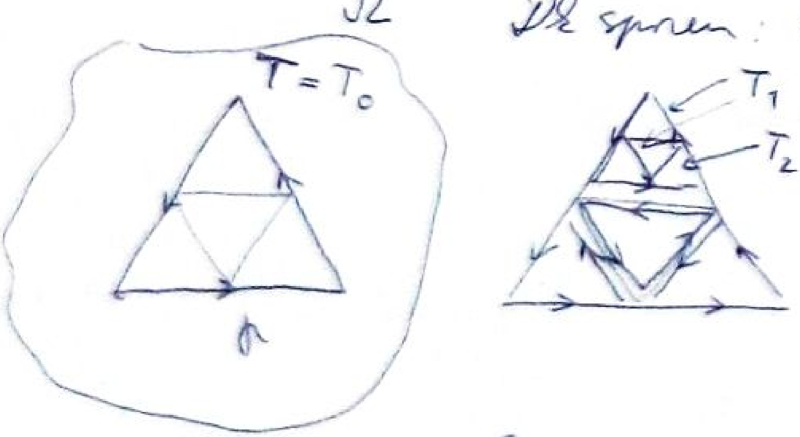
$\boxed{\text{Cauchyova věta a vzorec}}$

$\boxed{\text{hvězdovitá množina}}$



$\boxed{\text{K důkazu Cauchyovy věty}}$ Ať T je Δ v Ω a cl. celý vnitřek T leží v Ω. Ukážeme, že $\int_\gamma f(z)\,dz = 0$ pro $f \in \mathcal{H}(\Omega)$, γ ... hranice Δ T.



Dk sporem: ať $\left|\int_\gamma f(z)\,dz\right| > 0$ $=: K$

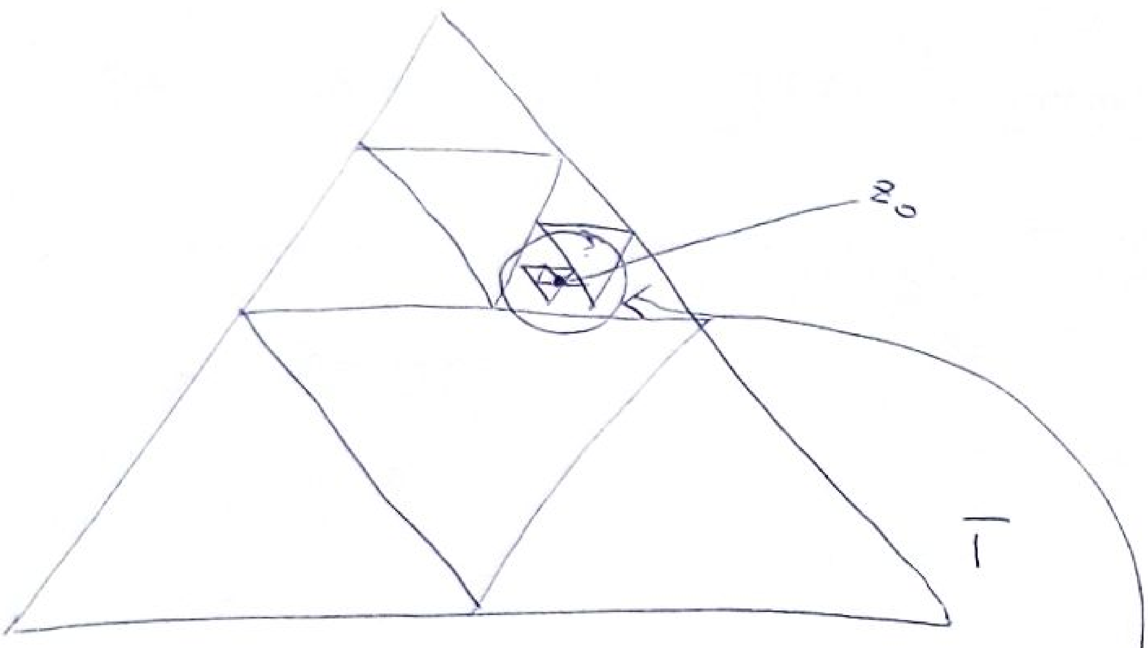
Ex. menší Δ takový, že ozn. T_1

$$\left|\int_{\partial T_1} f(z)\,dz\right| \geq \frac{K}{4}$$

$$\longrightarrow \{T_k\}_{k=0}^{+\infty} : \left|\int_{\partial T_k} f(z)\,dz\right| \geq \frac{K}{4^k}$$

$$\longrightarrow \left|\int_{\partial T_k} 1\,dz\right| = \frac{L}{2^k}, \quad L = \left|\int_{\partial T_0} 1\,dz\right|$$

$$\longrightarrow \bigcap_{k \in \mathbb{N}} \overline{T}_k = \{z_0\}$$



v z_0 má f komplexní derivaci

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + zb \quad \text{na okolí } z_0$$

Pro velká k

$$\int_{\partial T_k} f(z)\,dz = \int_{\partial T_k} f(z_0) + f'(z_0)(z-z_0) + zb\,dz$$

$$= \int_{\partial T_k} zb\,dz$$

$$|zb| = o(|z-z_0|) \text{ pro } z \to z_0$$

$$\le \varepsilon |z-z_0| \le \varepsilon \frac{D}{2^k}$$

$$\frac{K}{4^k} \le \int_{\partial T_k} f(z)\,dz = \int_{\partial T_k} o(|z-z_0|)\,dz$$

$$\le \varepsilon \frac{D}{2^k} \cdot \left|\int_{\partial T_k} 1\,dz\right| \le \varepsilon \frac{D}{2^k}\frac{L}{2^k}$$

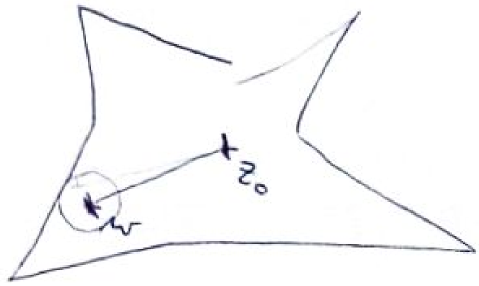
$$\le \varepsilon \frac{DL}{4^k}$$

↯ ε je lib. malé

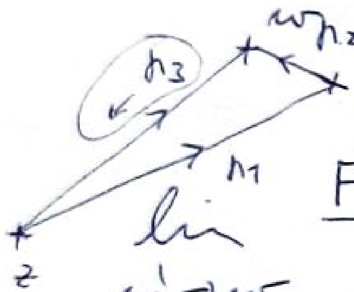
D průměr T_0

$\boxed{\text{Květe 3.5}}$ • ukážeme, že f má prim. fci

$$F(w) = \int_\gamma f(z)\,dz \text{, kde } z_0 \text{ je "střed hvězdy" a}$$
$$\gamma \text{ je úsečka } [z_0, w]$$



• F je PF $\Leftarrow$ z Cauchyovy věty pro Δ



$$F'(w)$$

$$\lim_{w' \to w} \frac{F(w') - F(w)}{w' - w} = (*)$$

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f = 0 = F(w') - F(w) + \int_{\gamma_2} f$$

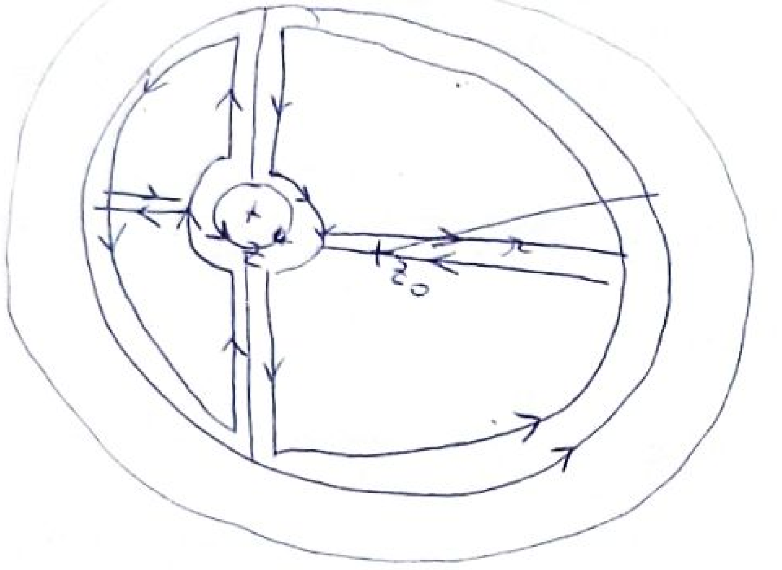
$$(*) = \lim_{w' \to w} \frac{-\int_{\gamma_2} f}{w' - w} = \lim_{w' \to w} - \frac{\int_0^1 f(\gamma_2(t))\cancel{(w - w')}}{\cancel{w' - w}} dt$$

$$\gamma_2(t) = w' + t(w - w'),\ t \in [0,1]$$

$$(*) = \lim_{w' \to w} \int_0^1 f(\gamma_2(t))\,dt = f(w) \checkmark$$

• f nemá na Ω PF $\Rightarrow \int_\varphi f = 0$

$\boxed{\text{Cauchyho vzorec}}$



1) $$f(z) \stackrel{?}{=} \frac{1}{2\pi i} \int_{\gamma_{z_0,r}} \frac{f(w)}{w-z}\, dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_\varepsilon} \frac{f(w)}{w-z}\, dw \quad ; \text{ kde } \gamma_\varepsilon(t) = z + \varepsilon e^{it},\ t \in [0, 2\pi]$$

$$= \frac{1}{2\pi \cancel{i}} \int_0^{2\pi} \frac{f(\gamma_\varepsilon(t))}{\cancel{\varepsilon e^{it}}} \cdot \cancel{\varepsilon i e^{it}}\, dt = \frac{1}{2\pi} \int_0^{2\pi} f(\gamma_\varepsilon(t))\, dt \xrightarrow[\varepsilon \to 0+]{} f(z)$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma_{z_0,1}} \frac{f(w)}{w-z}\, dw = f(z)$$

2) $$f'(z) = \frac{d}{dz}\left(\frac{1}{2\pi i} \int_{\gamma_{z_0,r}} \frac{f(w)}{w-z}\, dw \right) = \frac{1}{2\pi i} \int_{\gamma_{z_0,r}} \frac{f(w)}{(w-z)^2}\, dz$$

$$f''(z) = \frac{1}{2\pi i} \int_{\gamma_{z_0,r}} \frac{f(w)}{(w-z)^3} \cdot 1 \cdot 2\, dz$$

$$f^{(n)}(z) = \frac{1}{2\pi i} \int_{\gamma_{z_0,r}} \frac{f(w)}{(w-z)^{n+1}}\, 1 \cdot 2 \cdots n\, dz$$

$$e^t = \sum_{k=0}^{+\infty} \frac{t^k}{k!}$$

má střed konvergence $= 0$ $\boxed{v\ \mathbb{R}}$

pol. konvergence $= +\infty$

$$\lg(1-t) = -\sum_{k=1}^{+\infty} \frac{t^k}{k}$$ střed 0, pol. $= 1$

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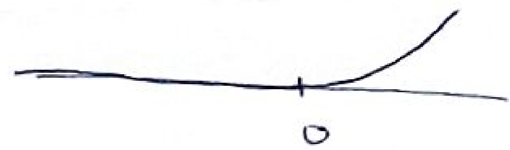
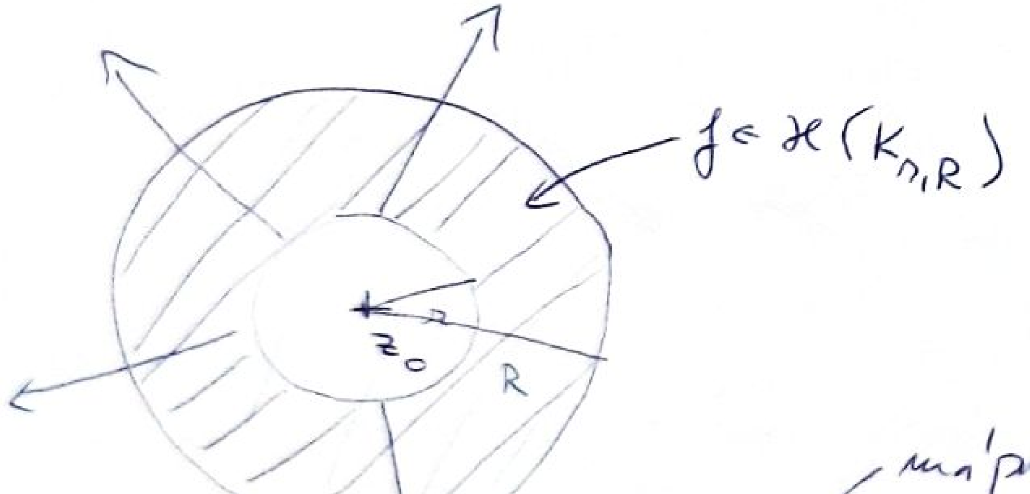
v $\mathbb{R}$ ex. fce, která je C^∞, ale není součtem své Taylorovy řady

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases}$$



pak $f \in C^\infty(\mathbb{R})$, $f^{(k)}(0) = 0$



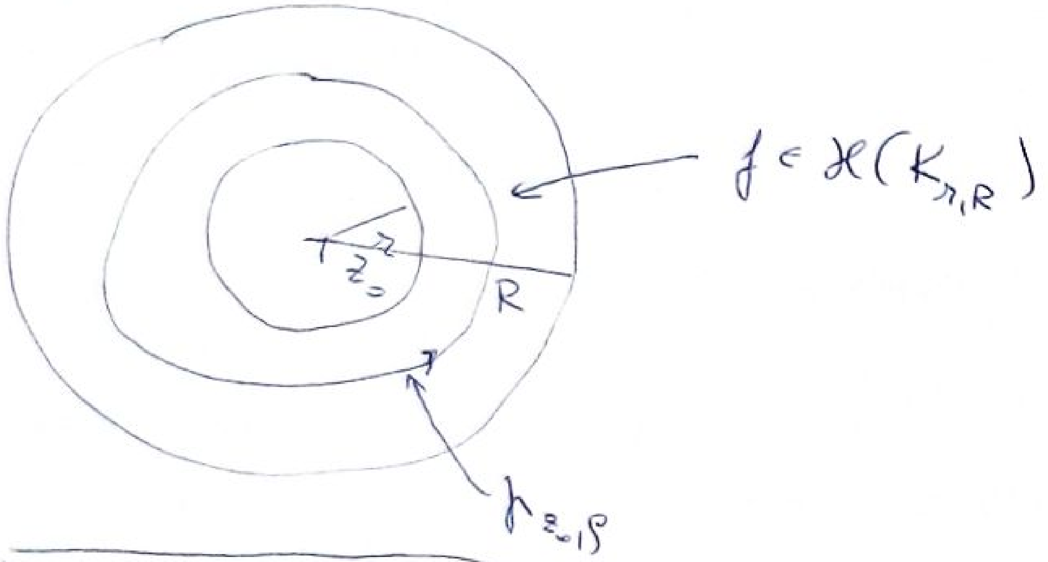
má pol. konv. R

$$f(z) = \sum_{n=-\infty}^{\cancel{+\infty}\ -1} a_n (z-z_0)^n + \boxed{\sum_{n=0}^{+\infty} a_n (z-z_0)^n}$$

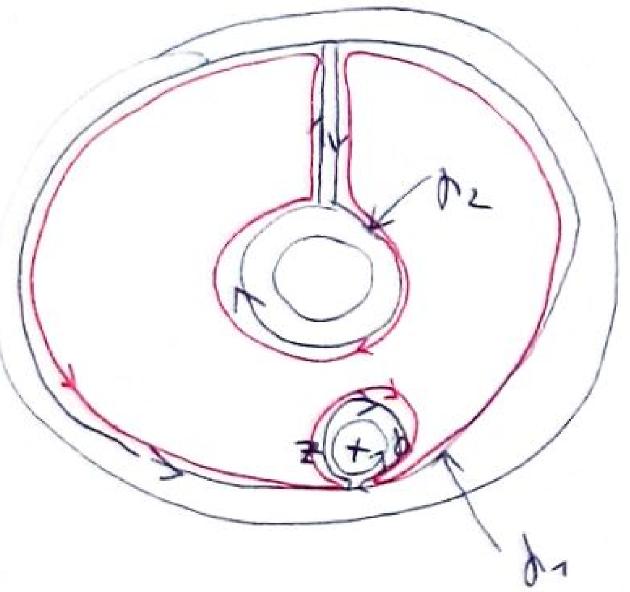
$$\|$$

$$\boxed{n = -m}$$

$$= \sum_{m=1}^{+\infty} a_{-m} \left(\frac{1}{z-z_0}\right)^m$$



$\boxed{\text{Veta 3.10 – násnad}}$



$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(w)}{w-z}\,dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_1+\gamma_2} \frac{f(w)}{w-z}\,dw$$

$$\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{w-z_0-(z-z_0)}\,dw$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{1-\underbrace{\left(\frac{z-z_0}{w-z_0}\right)}_{q,\ |q|<1}} \frac{1}{w-z_0}\,dw =$$

$$= \frac{1}{2\pi i} \int_{\gamma_1} \sum_{n=0}^{+\infty} f(w) \cdot \left(\frac{z-z_0}{w-z_0}\right)^n \frac{1}{w-z_0}\,dw$$

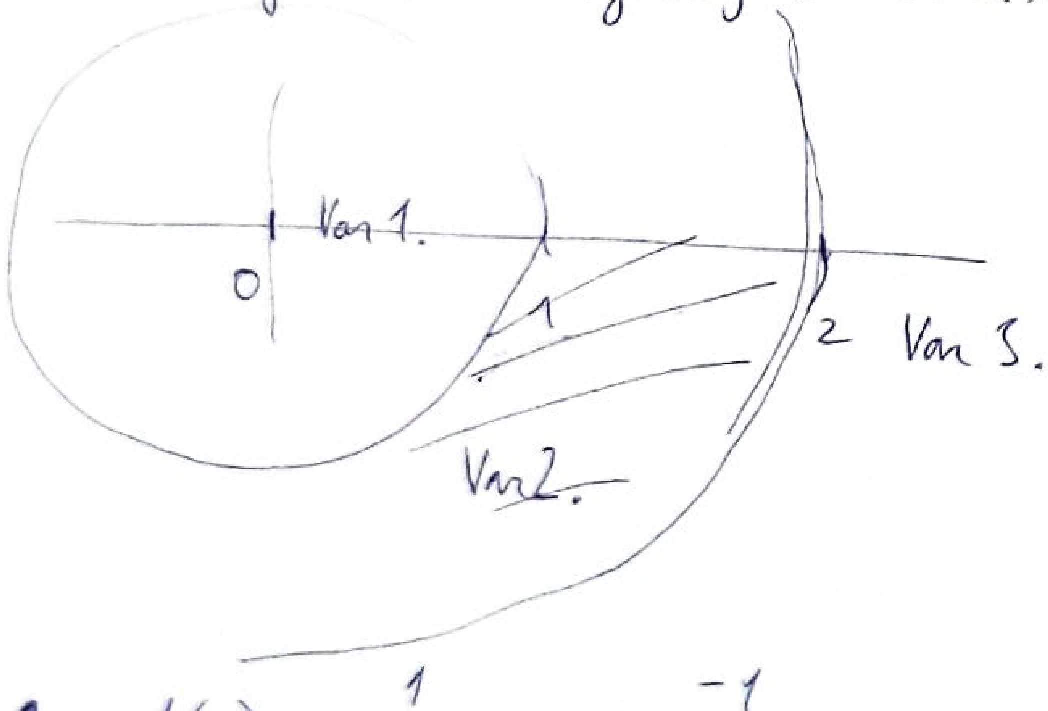
$$\overset{P}{=} \sum_{n=0}^{+\infty} (z-z_0)^n \underbrace{\frac{1}{2\pi i} \int_{\gamma_1} \frac{f(w)}{(w-z_0)^{n+1}}\, dw}_{a_n}$$

prohození
$\sum$ a int

Př: $f(z) := \dfrac{1}{z-1} + \dfrac{1}{z-2}$

$f \in \mathcal{H}(\mathbb{C} \setminus \{1,2\})$

Jak rozvinout f do Laurentovy řady se středem v 0?



Var 2: $f(z) = \dfrac{1}{z(1-\frac{1}{z})} + \dfrac{-1}{2(1-\frac{z}{2})} =$

$$= \frac{1}{z} \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^n - \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{z}{2}\right)^n = \sum_{n=0}^{+\infty} \left(\frac{1}{z}\right)^{n+1} - \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{z}{2}\right)^n$$

$$= \left| m = -(n+1) \right| = \sum_{m=-\infty}^{-1} z^m - \frac{1}{2} \sum_{n=0}^{+\infty} \left(\frac{z}{2}\right)^n$$

hlavní a regulární část