Algorithms and datastructures I
Lecture 7: tree based data-structures

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Kruskal algorithm, 1956

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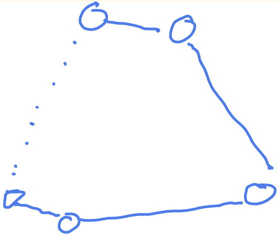
Input: Connected graph $G = (V, E)$ and weight function w with unique weights

1. Sort edges by weights; $w(e_1) \leq \cdots \leq w(e_m)$
2. $T \leftarrow (V, \emptyset)$
3. For $i = 1, \ldots m$:
4. $\quad u, v \leftarrow$ vertices in edge e_i
5. $\quad$ If u and v are in different components of T:
6. $\qquad T \leftarrow T + e_i$.

Output: Minimum spanning tree T.



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Theorem

Kruskal algorithm finds minimal spanning tree in time $O(m \log n + mT_f(n) + nT_u(n))$ where T_f is time complexity of $\text{F{\scriptsize IND}}$ *and T_u is a time complexity of* $\text{U{\scriptsize NION}}$ *on graph with n vertices.*

Union-find using arrays

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Kruskal algorithm finds minimal spanning tree in time $O(m \log n + mT_f(n) + nT_u(n))$ where T_f is time complexity of $\textsc{Find}$ *and T_u is a time complexity of* $\textsc{Union}$ *on graph with n vertices.*

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Idea: use array c. For a given vertex v put $c(v)$ to ID of a component it belongs to.

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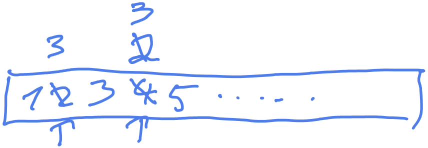
Array based union-find

FIND(u,v): $O(1)$ (return true compare if $c(u) = c(v)$)
UNION(u,v): $O(n)$ (search array v and change all occurrences of $c(u)$ to $c(v)$)



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Runtime of complete algorithm: $O(m\log n + m + n^2) = O(m\log n + n^2)$

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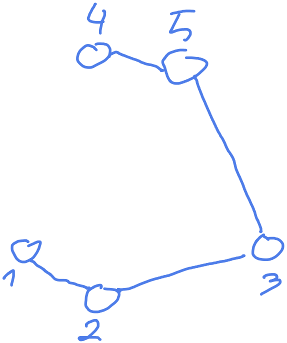
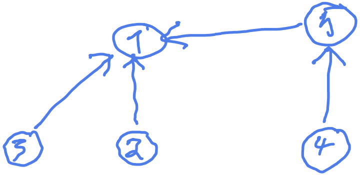
Runtime of complete algorithm: $O(m \log n + m + n^2) = O(m \log n + n^2)$

Homework: Try to analyze variant where you always rename the smaller component in time $O(s)$ where s is the size of the component. (it does improve time complexity).

Union-find with “Shrubs” (Gallner, Fisher 1964)





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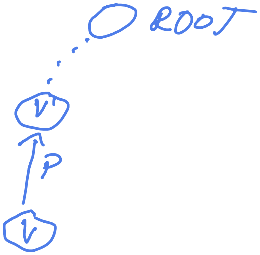
Root (v)

1. While $P(v) \neq \emptyset$:
2. $\quad v \leftarrow P(v)$
3. Return v.



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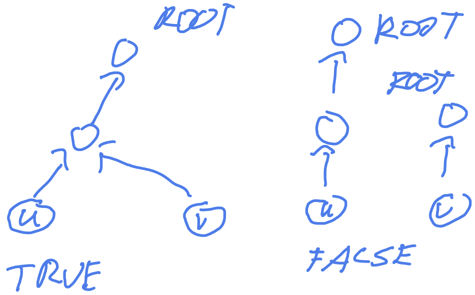
1. Return true if Root (u)=Root (v).



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Union (u, v)

1. $a \leftarrow$ Root (u), $b \leftarrow$ Root (v)
2. If $a = b$: return
3. $P(b) \leftarrow a$

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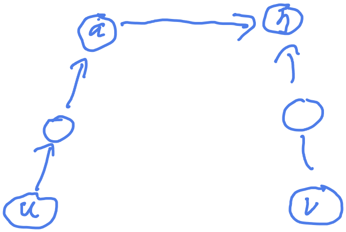

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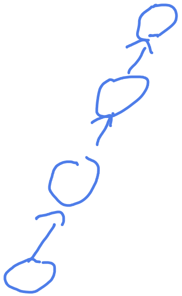
Runtime of Root(v) is bounded by the maximal height of a shrub.



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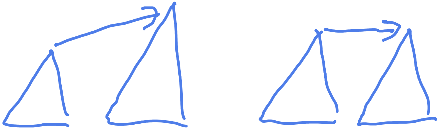

Union-find with “Shrubs” (Gallner, Fisher 1964)

Array P holds predecessor of a vertex (and $\emptyset$ for root). H on a root vertex holds the height of its tree.

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Union (u, v)

1. $a \leftarrow$ Root (u), $b \leftarrow$ Root (v)
2. If $a = b$: return
3. If $H(a) < H(b)$: $P(a) \leftarrow b$
4. If $H(a) > H(b)$: $P(b) \leftarrow a$
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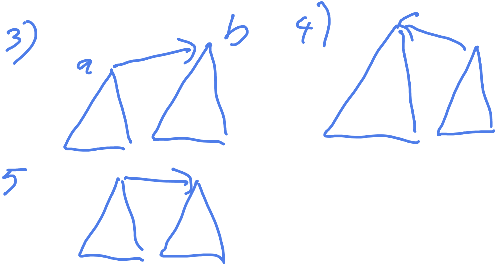

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Invariant

Shrub of height h has at least 2^h vertices

1) $h=0$ ✓

$h-1$ $\Rightarrow$ h

$2^{h-1} + 2^{h-1} = 2^h$ ✓

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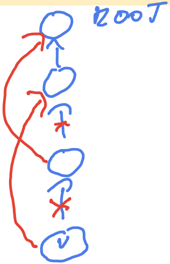
Theorem

Time complexity of UNION *and* FIND *is* $O(\log n)$.

Union-find with path compression

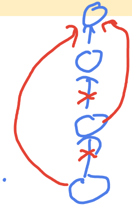
Root (v) with path compression variant 1

1. While $P(v) \neq \emptyset$:
2. $\quad u \leftarrow v$
3. $\quad v \leftarrow P(v)$
4. $\quad$ if $P(v) \neq \emptyset$ then:
5. $\qquad P(u) \leftarrow P(v)$
6. Return v.



Root (v) with path compression variant 2

1. $u \leftarrow v$
2. While $P(v) \neq \emptyset$:
3. $\quad v = P(v)$
4. While $P(u) \neq \emptyset$:
5. $\quad w \leftarrow P(u)$
6. $\quad P(u) \leftarrow v$
7. $\quad u \leftarrow w$
8. Return v.



Union-find with path compression

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Root (v) with path compression variant 2

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7. $\quad u \leftarrow w$
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In 1975 Robert Tarjan shown that adding the path compression reduces the time to $O(\alpha(n))$ where α is the inverse of Ackerman function.

$\cdots \log \log \log n$

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In 1975 Robert Tarjan shown that adding the path compression reduces the time to $O(\alpha(n))$ where α is the inverse of Ackerman function.
Ackerman function is very fast growing function. $A(4)$ is approximately

$$2^{2^{2^{2^{16}}}}$$

Thus we can think of it as an $O(1)$ implementation.

Set datastructure

We would like to represent a set (or a dictionary) of some elements from an universum.
We expect that elements of universum in set can be assigned and compared in $O(1)$

- INSERT(v): Insert v to the set
- DELETE(v): Delete v from the set
- FIND(v): Find v in the set
- SHOW: Print whole set
- MIN: Return minimum
- MAX: Return maximum
- SUCC(v): Find successor
- PRED(v): Find predecessor

$=$, $<$, $>$, $==$

ASSIGN   EQUAL

Basic implementations

	INSERT	DELETE	FIND	MIN/MAX	SUCC/PRED
Linked list	$O(n)$ or $O(1)$	$O(n)$ or $O(1)$	$O(n)$	$O(n)$	$O(n)$

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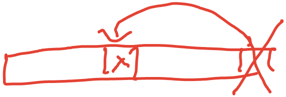

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Array	$O(n)$ or $O(1)$	$O(n)$ or $O(1)$	$O(n)$	$O(n)$	$O(n)$

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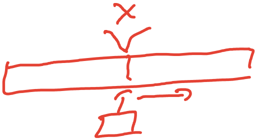

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Array	$O(n)$ or $O(1)$	$O(n)$ or $O(1)$	$O(n)$	$O(n)$	$O(n)$
Sorted array	$O(n)$	$O(n)$	$O(\log n)$	$O(1)$	$O(\log n)$ or $O(1)$

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- $\textsc{Insert}(v)$: Insert v to the set
- $\textsc{Delete}(v)$: Delete v from the set
- $\textsc{Find}(v)$: Find v in the set
- $\textsc{Show}$: Print whole set
- $\textsc{Min}$: Return minimum
- $\textsc{Max}$: Return maximum
- $\textsc{Succ}(v)$: Find successor
- $\textsc{Pred}(v)$: Find predecessor

Basic implementations

	Insert	Delete	Find	Min/Max	Succ/Pred
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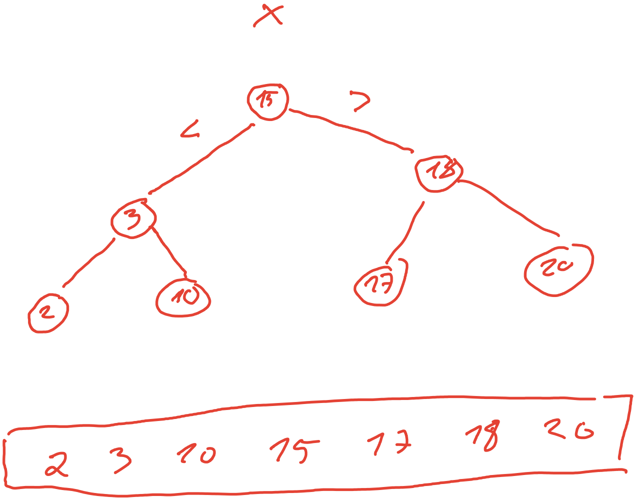
Sorted array	$O(n)$	$O(n)$	$O(\log n)$	$O(1)$	$O(\log n)$ or $O(1)$

Today: We design datastructure that does all in an logarithm.

Binary search trees

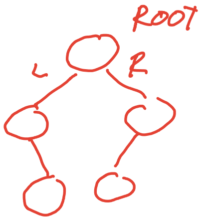


Binary search trees

Definition (Binary tree)

Binary tree is:

1. a rooted tree where
2. every vertex has at most 2 sons and
3. we where distinguish left and right son of every vertex



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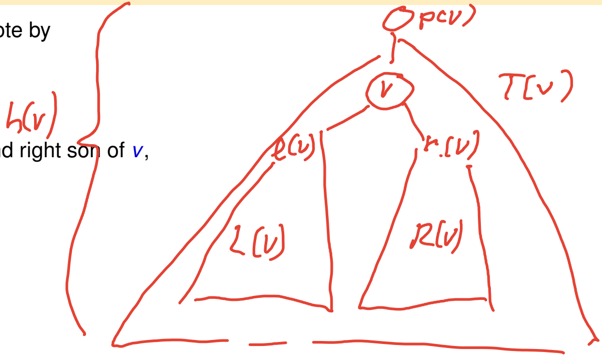
Notation: for a vertex v in a binary tree we denote by

- $l(v)$ and $r(v)$ the left and right son of v,
- $p(v)$ the parent of v.
- $T(v)$ the subtree rooted in v,
- $L(v)$ and $R(v)$ the subtree rooted in left and right son of v,
- $h(v)$ the height of $T(v)$.



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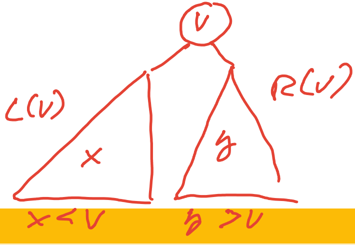


Definition (Binary search tree)

Binary search tree is a binary tree where every vertex v has unique key $k(v)$ and for every vertex v it holds:

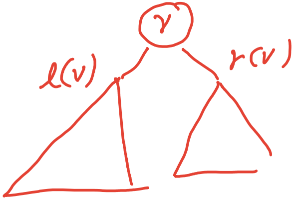
1. $\forall_{x \in L(v)} : x < v$ and
2. $\forall_{y \in R(v)} : y > v$.

keys

Operations on binary search trees

Show(v): Print all values in a tree with root v

1. If $v = \emptyset$: return
2. Show ($l(v)$)
3. Print v
4. Show ($r(v)$)



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Find(v,x): Find key x in a tree with root v

1. If $v = \emptyset$: return $\emptyset$
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3. If $x < k(v)$: return Find($l(v)$,x)
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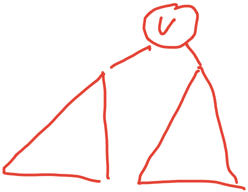

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Min(v): Return minimum of a tree with root v

1. If $v = \emptyset$: return $\emptyset$
2. If $l(v) = \emptyset$: return v
3. Return Min($l(v)$)

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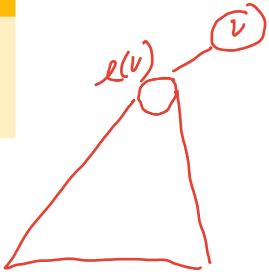

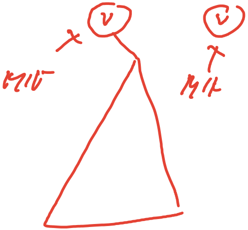

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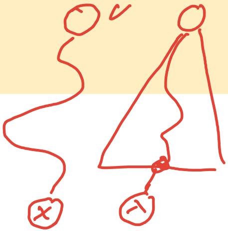
3. Return Min($l(v)$)



Find(v,x): Find key x in a tree with root v

1. If $v = \emptyset$: return $\emptyset$
2. If $x = k(v)$: return v
3. If $x < k(v)$: return Find($l(v)$,x)
4. If $x > k(v)$: return Find($r(v)$,x)

Insert(v,x): Insert x to a tree with root v

1. If $v = \emptyset$: create new vertex v with key x and return it
2. If $x < k(v)$: $l(v) \leftarrow$ Insert ($l(v)$,x)
3. If $x > k(v)$: $r(v) \leftarrow$ Insert ($r(v)$,x)
4. If $x = k(v)$: then x already exists in the tree and there is nothing to do.

Operations on binary search trees

Show(v): Print all values in a tree with root v

1. If $v = \emptyset$: return
2. Show ($l(v)$)
3. Print v
4. Show ($r(v)$)

Find(v,x): Find key x in a tree with root v

1. If $v = \emptyset$: return $\emptyset$
2. If $x = k(v)$: return v
3. If $x < k(v)$: return Find($l(v)$,x)
4. If $x > k(v)$: return Find($r(v)$,x)

Min(v): Return minimum of a tree with root v

1. If $v = \emptyset$: return $\emptyset$
2. If $l(v) = \emptyset$: return v
3. Return Min($l(v)$)

Insert(v,x): Insert x to a tree with root v

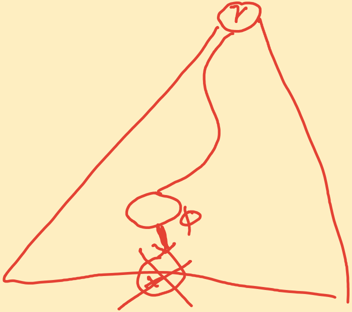
1. If $v = \emptyset$: create new vertex v with key x and return it
2. If $x < k(v)$: $l(v) \leftarrow$ Insert ($l(v)$,x)
3. If $x > k(v)$: $r(v) \leftarrow$ Insert ($r(v)$,x)
4. If $x = k(v)$: then x already exists in the tree and there is nothing to do.

Homework: Figure out implementation of SUCC and PRED

Delete in binary search tree

Delete(v,x): Insert x to a tree with root v

1. If $v = \emptyset$: return $\emptyset$
2. If $x < k(v)$: $l(v) \leftarrow$Delete($l(v)$,x)
3. If $x > k(v)$: $r(v) \leftarrow$Delete($r(v)$,x)
4. If $x = k(v)$:
5. If $l(v) = r(v) = \emptyset$: return $\emptyset$



Delete in binary search tree

Delete(v,x): Insert x to a tree with root v

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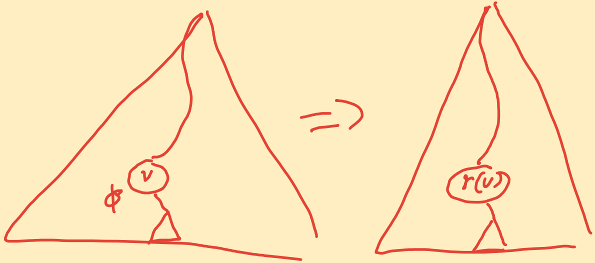
2. If $x < k(v)$: $l(v) \leftarrow$Delete($l(v)$,x)
3. If $x > k(v)$: $r(v) \leftarrow$Delete($r(v)$,x)
4. If $x = k(v)$:
5. If $l(v) = r(v) = \emptyset$: return $\emptyset$
6. If $l(v) = \emptyset$: return $r(v)$



Delete in binary search tree

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1. If $v = \emptyset$: return $\emptyset$
2. If $x < k(v)$: $l(v) \leftarrow$Delete($l(v)$,x)
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4. If $x = k(v)$:
5. If $l(v) = r(v) = \emptyset$: return $\emptyset$
6. If $l(v) = \emptyset$: return $r(v)$
7. If $r(v) = \emptyset$: return $l(v)$

Delete in binary search tree

Delete(v,x): Insert x to a tree with root v

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2. If $x < k(v)$: $l(v) \leftarrow$Delete($l(v)$,x)
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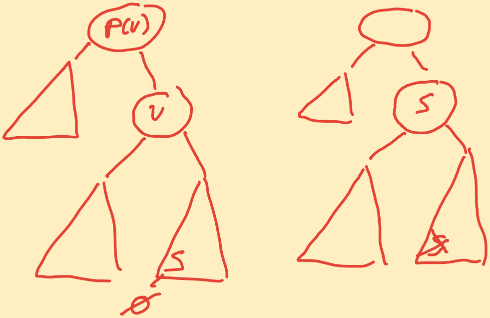
4. If $x = k(v)$:
5. If $l(v) = r(v) = \emptyset$: return $\emptyset$
6. If $l(v) = \emptyset$: return $r(v)$
7. If $r(v) = \emptyset$: return $l(v)$
8. $s \leftarrow$Min($\cancel{v}$) $\leftarrow r(v)$
9. $k(v) \leftarrow k(s)$
10. $r(v) \leftarrow$Delete($r(v)$,s)



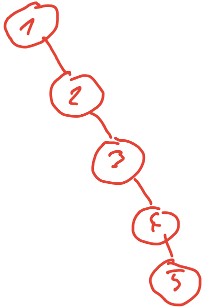
Time complexity

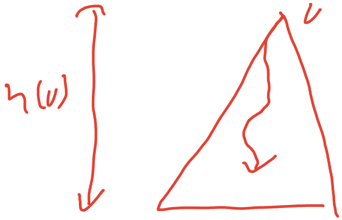
Theorem

Operations INSERT, DELETE, FIND, MIN, MAX, SUCC *and* PRED *on binary search tree runs in time* $O(h)$ *where* h *is a height of the tree.*

Sadly the height of a binary search tree can be n.

1, 2, 3, 4, 5 .....





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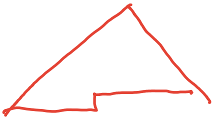
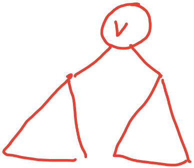
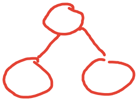
Definition (Perfectly ballanced tree)

Binary search tree is perfectly balanced if $\forall_v : \big| |L(v)| - |R(v)| \big| \leq 1$.

Depth of perfectly balanced tree is $\lfloor \log n \rfloor$.







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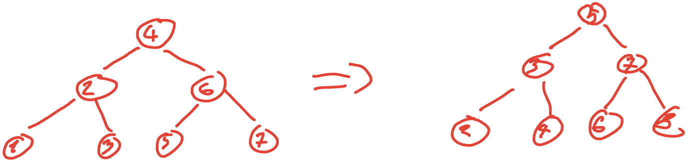
Theorem

The time complexity of insert on perfectly balanced tree is $\Omega(n)$.

Put $n = 2^k - 1$ and then perform Insert(1), Insert (2),. . . , Insert(n).
Continue by Delete(1), Insert$(n+1)$, Delete(2), Insert$(n+2)$, . . .



$\Omega(n)$

AVL-trees (1962)

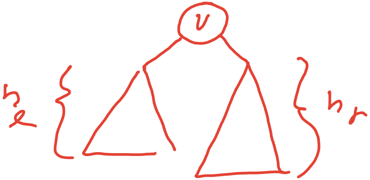
Georgy Adelson-Velsky

Evgenii Landis

Definition (AVL tree)

Binary search tree is height balanced (or AVL-tree) if

$$\forall_v : \left| h(l(v)) - h(r(v)) \right| \leq 1.$$



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Every AVL-tree with n vertices has depth $\Theta(\log n)$

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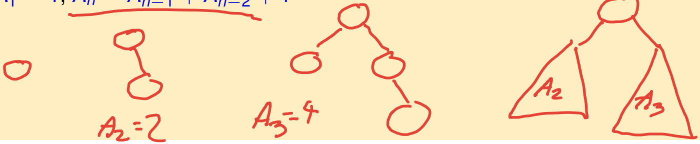
Proof.

Denote by A_n the minimal number of vertices of an AVL-tree. OF HEIGHT n
Show that $A_0 = 0$, $A_1 = 1$, $A_n = A_{n-1} + A_{n-2} + 1$



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Observe $A_n \geq 2^{\frac{n}{2}}$:

$$A_n = A_{n-1} + A_{n-2} \cancel{+1} \geq 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}}$$

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$$A_n = A_{n-1} + A_{n-2} + 1 \geq 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}} = 2^{\frac{n}{2}}\left(2^{-\frac{1}{2}} + 2^{-1}\right)$$

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$$A_n = A_{n-1} + A_{n-2} + 1 \geq 2^{\frac{n-1}{2}} + 2^{\frac{n-2}{2}} = 2^{\frac{n}{2}} \left(2^{-\frac{1}{2}} + 2^{-1}\right) > 2^{\frac{n}{2}} \overbrace{(0.707 + 0.5)}^{>1} > 2^{\frac{n}{2}}.$$

□

Insert operation

Remember for every vertex a sign $\delta(v) = h(l(v)) - h(r(v))$

Insert(v,x)

1. Insert element to a binary search tree
2. Re-balance the tree

log(n)

Insert case — —

Insert case $-+$