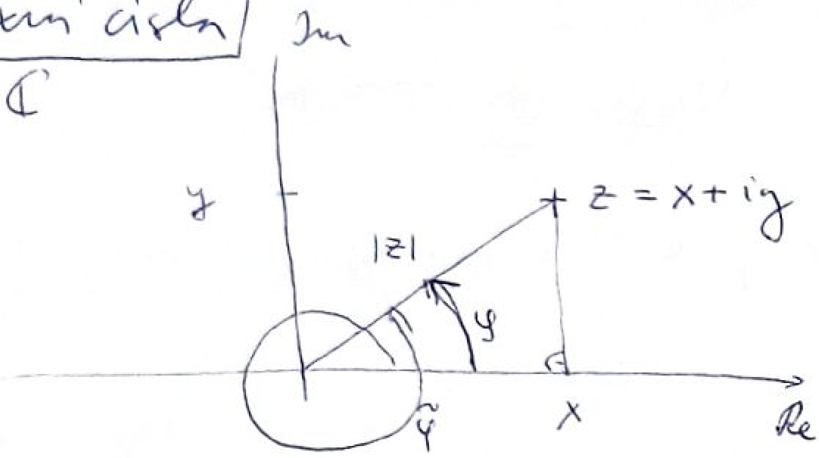
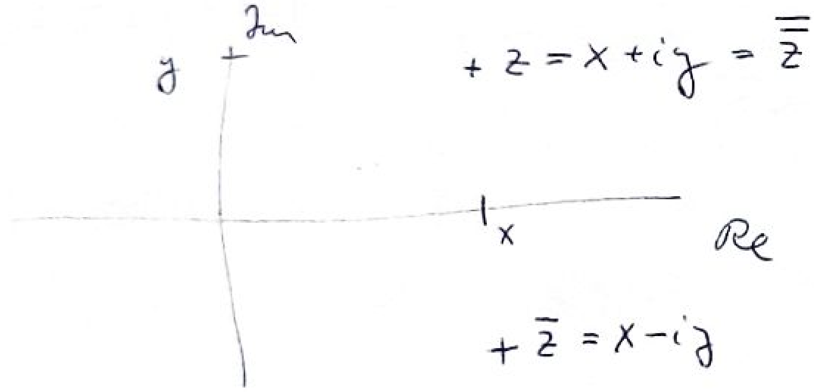
## Komplexní čísla

$\mathbb{C}$




$$|e^{i\varphi}| = |\cos\varphi + i\sin\varphi| = \sqrt{\cos^2\varphi + \sin^2\varphi} = 1$$

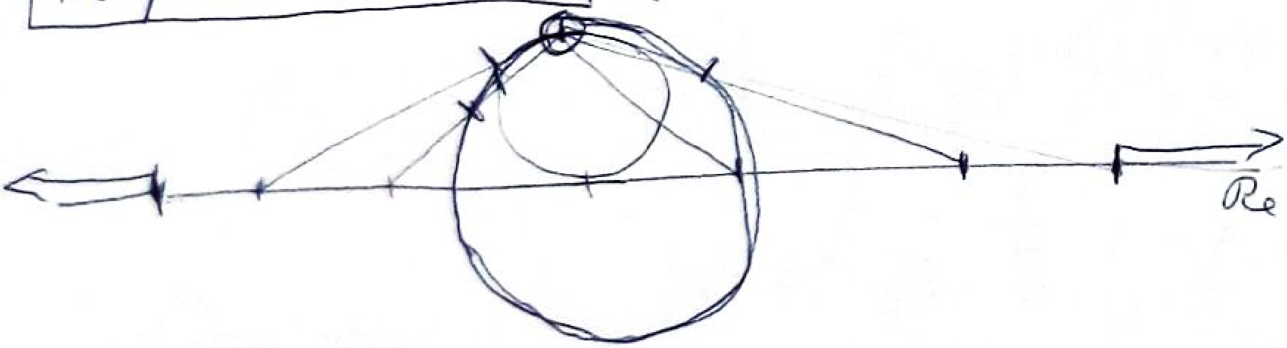
$$|z| = \sqrt{x^2+y^2}$$




$$z\cdot\bar{z} = (x+iy)(x-iy) = x^2 - \cancel{ixy} + \cancel{iyx} - i^2y^2 = x^2+y^2$$
$$= |z|^2$$

$$i^2 = -1$$

## Komplexní nekonečno $\infty$



## Limity

$f: \mathbb{C} \longrightarrow \mathbb{C} \qquad (f_1, f_2): \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$\longrightarrow$ limity a spojitost pro $f: \mathbb{C} \to \mathbb{C}$ jsou stejné jako pro $(f_1, f_2): \mathbb{R}^2 \to \mathbb{R}^2$

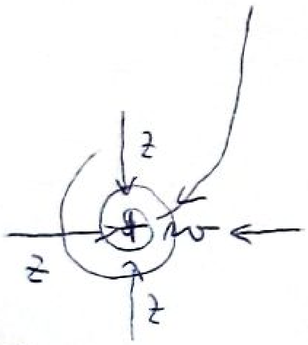
$f(z) = z$ je spojitá v $i$

$$\lim_{z \to i} f(z) = A$$

$$\forall \varepsilon > 0, \exists \delta > 0, 0 < |z - i| < \delta \Rightarrow |f(z) - A| < \varepsilon$$

$$|z - A| < \varepsilon$$

$$A = i \; ; \; \varepsilon =: \delta$$




## Derivace

$f(z) = z, \; f: \mathbb{C} \longrightarrow \mathbb{C}, \quad f(x+iy) = x + iy$

$$f_1(x, y) = x$$

$$f_2(x, y) = y$$

$$\frac{\partial f_1}{\partial x}(x,y) = 1 \overset{?}{=} \frac{\partial f_2}{\partial y}(x,y) = 1$$

$$\frac{\partial f_1}{\partial y}(x,y) = 0 \overset{?}{=} \frac{\partial f_2}{\partial x}(x,y) = 0$$

CR podm. platí!?

$$f'(z) = 1$$

$$f(z) := e^z \quad , \quad f(x+iy) = e^{x+iy} = e^x(\cos y + i \sin y)$$

$$f_1(x,y) = e^x \cos y$$

$$f_2(x,y) = e^x \sin y$$

$$\frac{\partial f_1}{\partial x}(x,y) = e^x \cos y \overset{!}{=} \frac{\partial f_2}{\partial y}(x,y) = e^x \cos y$$

$$\frac{\partial f_1}{\partial y}(x,y) = e^x(-\sin y) \overset{!}{=} - \frac{\partial f_2}{\partial x}(x,y) = - e^x \sin y$$

$$f'(z) = (e^z)' = e^x \cos y + i\, e^x \sin y = e^z$$

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$$f(z) = |z|^2 \quad , \quad f(x+iy) = x^2 + y^2$$

$$f_1(x,y) = x^2 + y^2$$

$$f_2(x,y) = 0$$

$$\frac{\partial f_1}{\partial x}(x,y) = 2x \neq \frac{\partial f_2}{\partial y}(x,y) = 0 \quad \text{pro } x \neq 0$$

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Jak dostaneme CR podm.: v $w \in \mathbb{C}$

- ať všechny der. ex. ex. $\lim\limits_{z \to w} \frac{f(z) - f(w)}{z - w}$

$z := w + t \; ; \; t \in \mathbb{R}$ : $\lim\limits_{t \to 0} \frac{f(w+t) - f(w)}{t} =$

$\cancel{\lim\limits_{t \to 0} f_1(w_1 + t\, i w_2) - f_1(}$

$$= \lim_{t \to 0} \frac{f_1(w_1 + t, w_2) - f_1(w_1, w_2)}{t} + i \frac{\left(f_2(w_1 + t, w_2) - f_2(w_1, w_2)\right)}{t}$$

$$= \frac{\partial f_1}{\partial x}(w_1, w_2) + i \frac{\partial f_2}{\partial x}(w_1, w_2)$$

$z := w + it$

$$\lim_{t\to 0} \frac{f(w+it) - f(w)}{it} =$$

$$\lim_{t\to 0} \frac{f_1(w_1, w_2+t) - f_1(w_1, w_2)}{it} + i\,\frac{f_2(w_1, w_2+t) - f_2(w_1, w_2)}{it}$$

$$= \frac{1}{i}\,\frac{\partial f_1}{\partial y}(w_1, w_2) + \frac{\partial f_2}{\partial y}(w_1, w_2)$$

$$\frac{\partial f_1}{\partial x} + i\,\frac{\partial f_2}{\partial x} = \underbrace{\frac{1}{i}}_{-i}\,\frac{\partial f_1}{\partial y} + \frac{\partial f_2}{\partial y} \qquad \frac{1}{i} = \frac{(-1)\,i\cdot i}{i} = -i$$

$$\Rightarrow \boxed{\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial f_2}{\partial y} \\ \frac{\partial f_2}{\partial x} &= -\frac{\partial f_1}{\partial y} \end{aligned}}$$

Cauchy - Riemannovy podmínky

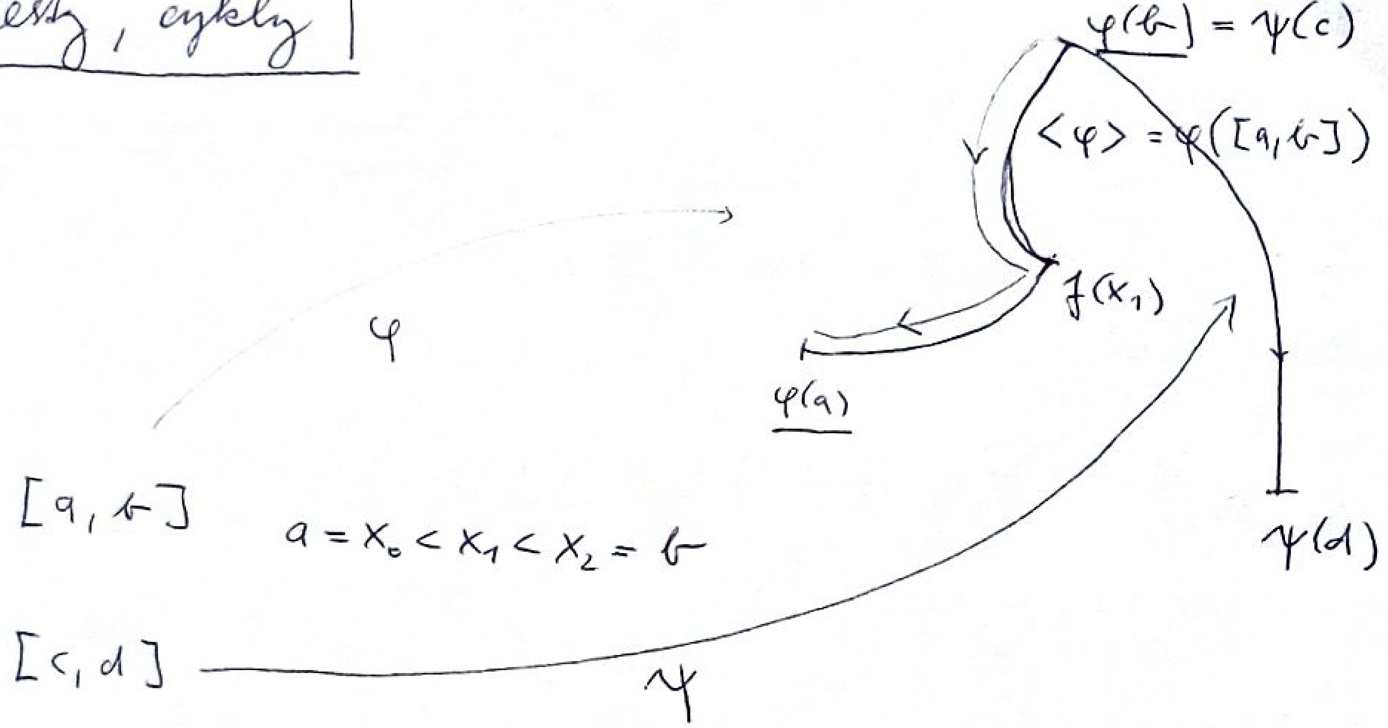
$f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ ; $f(x,y)$ $\quad \Delta f(x,y) = \frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y)$

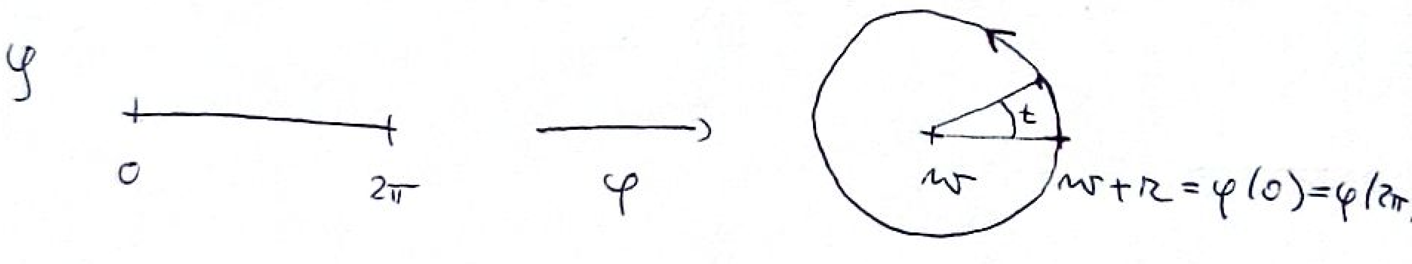
$$\frac{\partial^2 f_1}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f_1}{\partial x}\right) \overset{CR}{=} \frac{\partial}{\partial x}\left(\frac{\partial f_2}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f_2}{\partial x}\right) =$$

$$\overset{CR}{=} \frac{\partial}{\partial y}\left(-\frac{\partial f_1}{\partial y}\right) \Rightarrow \underline{\Delta f_1 = 0}$$

$\Delta_{xy}(e^x \cos y) = 0$

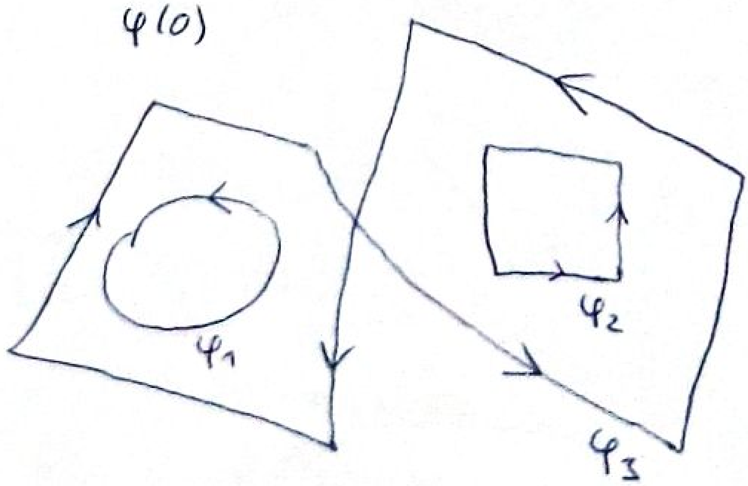
# Cesty, cykly





Cykl:



$\varphi(t) := e^{it}$ ; $[0, 2\pi]$ $\quad \varphi'(t) = i e^{it}$

$f(z) := \frac{1}{z}$

$\boxed{\text{Křivkový integrál v } \mathbb{C}}$

$$\int_\varphi f = \int_0^{2\pi} \frac{1}{\cancel{e^{it}}} \cdot i \cancel{e^{it}} \, dt = 2\pi i$$

$\int_\varphi \frac{1}{z^k} dz = 0$, pokud $k \in \mathbb{Z} \setminus \{1\}$.

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$$\left|\int_\varphi f\right| = \left|\int_a^b f\circ\varphi \cdot \varphi'\right| \leq \int_a^b |f\circ\varphi| \cdot |\varphi'| \leq \max_{\langle\varphi\rangle} |f| \underbrace{\int_a^b |\varphi'|}_{L(\varphi)}$$

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$$\int_\varphi f = \int_a^b f\circ\varphi \cdot \varphi' = \int_a^b F'\circ\varphi \cdot \varphi' = \int_a^b (F\circ\varphi)' = [F\circ\varphi]_a^b$$

$$= F(\varphi(b)) - F(\varphi(a))$$

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Jak def. PF?

$F(z) = \int_\varphi f$, kde $\varphi: z_0 \to z$

Př: $f(z) = \frac{1}{z}$ nemá PF na $U(0,2) \setminus \{0\}$

$\int_\varphi f = 2\pi i$; $\varphi(t) = e^{it}$, $t \in [0, 2\pi]$