Opakování

$x \in X$; $x = \sum c_n e_n$; $c_n = \frac{(x, e_n)}{\|e_n\|^2}$; $\|e_n\|^2 = (e_n, e_n)$

$L^2_\varrho(a,b)$ ... $(f,g) := \int_a^b \varrho f \bar{g}$; $\|f\| = \sqrt{\int_a^b \varrho |f|^2}$

$z = a + ib$; $\bar{z} = a - ib$

$t \in (a,b)$

$g : (a,b) \to \mathbb{C}$; $g(t) := g_1(t) + i g_2(t)$; $\bar{g}(t) := g_1(t) - i g_2(t)$

Př 5: Legendreovy polynomy

$\tilde{P}_0(x) = 1$; $\tilde{P}_1(x) = x - \frac{(x,1)}{\|1\|^2} = x$

$(x,1) = \int_{-1}^{1} x \, dx = 0$

$\tilde{P}_2(x) = x^2 - \frac{(x^2,1)}{\|1\|^2} - \cancel{\frac{(x^2,x)}{\|x\|^2}x}$

$(x^2,x) = \int_{-1}^{1} x^2 \cdot x \, dx = 0$; $(x^2,1) = \int_{-1}^{1} x^2 dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}$

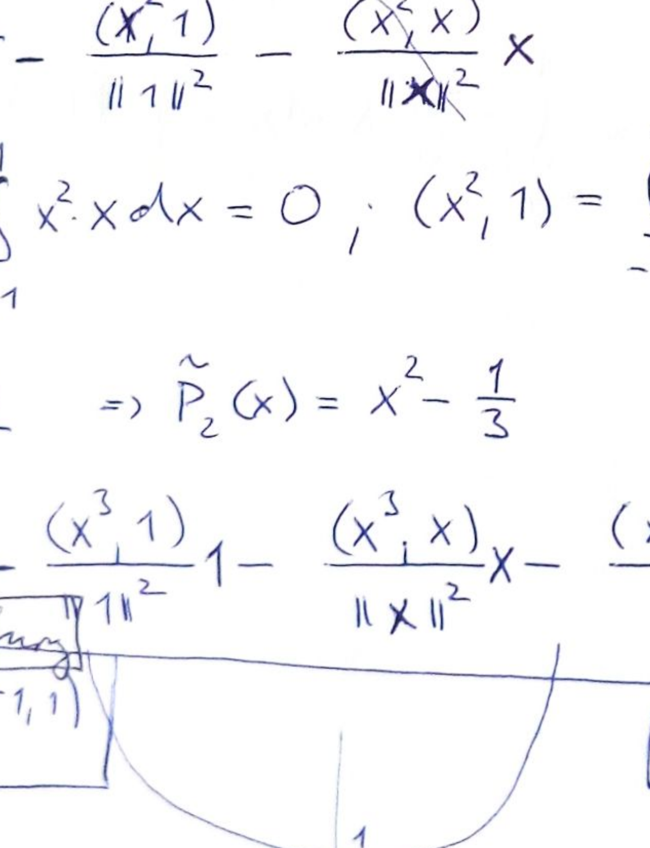
$\|1\|^2 = \int_{-1}^{1} 1 \, dx = 2 \Rightarrow \tilde{P}_2(x) = x^2 - \frac{1}{3}$

$\tilde{P}_3(x) = x^3 - \frac{(x^3,1)}{\|1\|^2} 1 - \frac{(x^3,x)}{\|x\|^2} x - \frac{(x^3, x^2 - \frac{1}{3})}{\|x^2 - \frac{1}{3}\|^2}\left(x^2 - \frac{1}{3}\right)$

Př 6: Čebyševovy polynomy

$L^2_{\frac{1}{\sqrt{1-x^2}}}(-1,1)$

$\frac{1}{\sqrt{1-x^2}}$



$\|f\|^2 = \int_{-1} \frac{1}{\sqrt{1-x^2}} |f|^2 dx$

$L^2_{\frac{1}{\sqrt{1-x^2}}}(-1,1) \subset L^2(-1,1)$

$\|f\|^2 = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} |f|^2(x) \, dx$

$$L^2_{\frac{1}{\sqrt{1-x^2}}}(-1,1) \subset L^2(-1,1)$$

$\boxed{\text{Čebyševovy polynomy}}$

$$f \in L^2_{\frac{1}{\sqrt{1-x^2}}}(-1,1) \Rightarrow \int_{-1}^{1} |f|^2 = \int_{-1}^{1} \underbrace{|f|^2(x)\frac{1}{\sqrt{1-x^2}}}_{\geq 0} \cdot \underbrace{\sqrt{1-x^2}}_{\in[0,1]}\, dx$$

$$\leq \int_{-1}^{1} |f|^2 \frac{1}{\sqrt{1-x^2}}\, dx < +\infty \Rightarrow f \in L^2(-1,1)$$

$$\tilde{T}_0(x) = 1$$

$$\tilde{T}_1(x) = x - \frac{(x,1)}{\|1\|} \cdot 1 = x$$

$$\tilde{T}_2(x) = x^2 - \frac{(x^2,1)}{\|1\|^2} \cdot 1 - \frac{(x^2, \tilde{T}_1(x))}{\|\tilde{T}_1\|^2} \cdot \tilde{T}_1 = \dots$$

$$\tilde{T}_1 : \quad (x,1) = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} \cdot x\, dx = \left[-\sqrt{1-x^2}\right]_{-1}^{1} = 0$$

$\boxed{\text{Motivace k sekci 2.3 – Metoda II}}$

A symetrická matice, $\exists H$ matice: D diagonální $AH = HD$

D jordanův tvar matice A ; $H^T = H^{-1}$

$$H = \begin{pmatrix} | & & | \\ v_1 & \dots & v_d \\ | & & | \end{pmatrix} \quad ; \quad D = \begin{pmatrix} d_1 & & \emptyset \\ & \ddots & \\ \emptyset & & d_d \end{pmatrix}$$

d_j ... vlastní čísla

v_j ... vlastní vektory A

$$H \cdot H^T = I_d = \begin{pmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{pmatrix} \Rightarrow \boxed{(v_k, v_j) = \delta_{kj}}$$

$$\| \qquad H^T \cdot H \qquad \|$$

$$\begin{pmatrix} - & v_1 & - \\ & \vdots & \\ - & v_d & - \end{pmatrix} \begin{pmatrix} | & & | \\ v_1 & \dots & v_d \\ | & & | \end{pmatrix} = \begin{pmatrix} 1 & & \emptyset \\ & \ddots & \\ \emptyset & & 1 \end{pmatrix} \Rightarrow \{v_j\}_{j=1}^{d} \text{ je ON báze } \mathbb{R}^d$$

Použití: $v \in \mathbb{R}^d$; $v = \sum_{j=1}^{d} \alpha_j v_j$

$$Av = A \sum_j \alpha_j v_j = \sum \alpha_j A v_j = \sum_j \alpha_j d_j v_j$$

v bázi $\{v_j\}$ $v \sim (\alpha_1, \dots, \alpha_d)$

$$A : (\alpha_1, \dots, \alpha_d) \longrightarrow (d_1 \alpha_1, \dots, \alpha_d d_d)$$

Analogie: matice versus Věta 2.7

A symetrická matice

$$Av = \lambda v$$

$\rightsquigarrow$ λ_j vl. čísla

v_j vl. vektory

báze v $\mathbb{R}^d$

$$-(py')' + qy = \lambda \varrho y$$

+ okr. podm.

$\longrightarrow$ $\{\lambda_n\}_{n=1}^{\infty}$

$\{y_n\}_{n=1}^{+\infty}$

úplný OG systém v L^2_ϱ

Čebyševovy polynomy: $-\left(\sqrt{1-x^2}\,y'\right)' = +\dfrac{\lambda y}{\sqrt{1-x^2}}$

$p(x)$ $\quad$ ϱ - váha

Legendre: $-\left((1-x^2)\,y'\right)' = +\lambda y$

$=p(x)$

$\boxed{\text{Příklad na aplikaci Věty 2.7}}$

$a = 0;\ b = \pi,\ p, \varrho = 1,\ q = 0,\ \alpha, \gamma = 1,\ \beta, \delta = 0$

$$\Rightarrow \quad \left[\begin{array}{ll} -y'' = \lambda y & \text{v } (0,\pi) \\ y(0) = 0 = y(\pi) & \qquad L^2(0,\pi) \end{array}\right.$$

$y'' + \lambda y = 0 \qquad$ char. rce: $\alpha^2 + \lambda = 0 \Rightarrow \alpha_{1,2} = \pm\sqrt{-\lambda}$

a) $\lambda < 0;\ \lambda = -\mu^2;\ \mu > 0:\ \alpha_{1,2} = \pm\mu$

f.s.: $\{e^{\mu x};\ e^{-\mu x}\}$ obecné řešení: $A e^{\mu x} + B e^{-\mu x} = y(x)$

$$\left.\begin{array}{l} 0 = y(0) = A + B \\ 0 = y(\pi) = A e^{\mu\pi} + B e^{-\mu\pi} \end{array}\right\} \Rightarrow A, B = 0 \text{ protože } \det\begin{pmatrix} 1 & 1 \\ e^{\mu\pi} & e^{-\mu\pi} \end{pmatrix} \neq 0$$

b) $\lambda = 0$: neex. řešení

c) $\lambda > 0;\ \lambda = \mu^2,\ \mu > 0:\ \alpha_{1,2} = \pm i\mu$

f.s. $\{\sin(\mu x), \cos(\mu x)\}$ obecné řešení: $A \sin\mu x + B\cos\mu x = y(x)$

$0 = y(0) = B,\quad 0 = y(\pi) = A\sin(\mu\pi)$

$\Rightarrow \mu \in \mathbb{N}$

$\lambda_n = n^2 \quad y_n(x) = \sin(nx)$

$\Rightarrow \quad \{\sin(nx)\}_{n=1}^{+\infty}$ tvoří OG úplný systém v $L^2(0,\pi)$

V2.7

$\longrightarrow \{1, \sin(nx), \cos(nx)\}_{n=1}^{+\infty}$ úplný OG systém na $L^2(-\pi,\pi)$