TO BE IS TO BE A VALUE OF A VARIABLE (OR TO BE SOME VALUES OF SOME VARIABLES)*

ARE quantification and cross reference in English well represented by the devices of standard logic, i.e., variables $x,y,z,$ . . . , the quantifiers $\forall$ and $\exists$, the usual propositional connectives, and the equals sign? It's my impression that many philosophers and logicians think that—on the whole—they are. In fact, I suspect that the following view of the relation between logic and quantificational and referential features of natural language is fairly widely held:

No one (the view begins) can think that the propositional calculus contains all there is to logic. Because of the presence in natural language of quantificational words like 'all' and 'some' and words used extensively in cross reference, like 'it', 'that', and 'who', there is a vast variety of forms of inference whose validity cannot be adequately treated without the introduction of variables and quantifiers, or other devices to do the same work. Thus everyone will concede that the predicate calculus is at least a part of logic.

Indispensable to cross reference, lacking distinctive content, and pervading thought and discourse, *identity* is without question a logical concept. Adding it to the predicate calculus significantly increases the number and variety of inferences susceptible of adequate logical treatment.

And now (the view continues), once identity is added to the predicate calculus, there would not appear to be all that many valid inferences whose validity has to do with cross reference, quantification, and generalization which cannot be treated in a satisfactory way by means of the resulting system. It may be granted that there are certain valid inferences, involving so-called "analytic" connections, which cannot be handled in the predicate calculus with identity. But the validity of these inferences has nothing to do with quantification in natural language, and it may thus be doubted whether a logic that does nothing to explain their validity is thereby deficient.

In any event (the view concludes), the variety of inferences that

* I am grateful to Richard Cartwright, Helen Cartwright, James Higginbotham, Judith Thomson, and the editors of the *Journal of Philosophy* for helpful comments, criticism, and discussion. Helen Cartwright's valuable unpublished Ph.D. dissertation, "Classes, Quantities, and Non-singular Reference" (University of Michigan, 1963) deals at length with many of the issues with which the present paper is concerned.

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cannot be dealt with by first-order logic (with identity) is by no means as great or as interesting as the variety that can be handled by the predicate calculus, even without identity, but not by the propositional calculus.

It is the conclusion of this view that I want to take exception to. (At one time I thought the whole view was probably true.) It seems to me that we really do not know whether there is much or little in the province of logic that the first-order predicate calculus with identity cannot treat. In the first part of this paper I shall present and discuss some data which suggest that there may be rather more than might be supposed, that there may be an interesting variety both of quantificational and referential constructions in natural language that cannot be represented in standard logical notation and of valid inferences for whose validity these constructions are responsible. Whether quantification and cross reference in English are well represented by standard logic seems to me to be an open question, at present.

Several kinds of constructions, sentences, and inferences that cannot be symbolized in first-order logic are known. Perhaps the best-known of these involve numerical quantifiers such as 'more', 'most', and 'as many', e.g., the inference

Most Democrats are left-of-center.
Most Democrats dislike Reagan.
Therefore, some who are left-of-center dislike Reagan.

Another is the construction "For every *A* there is a *B*," which, although it might appear to be symbolizable in first-order notation, cannot be so represented, for it is synonymous with "There are at least as many *B*s as *A*s."[1] The construction is not of recent date; it is exemplified in a couplet from 1583 by one T. Watson:[2]

For every pleasure that in love is found,
A thousand woes and more therein abound.

Jaakko Hintikka has offered a number of examples of sentences that cannot, he claims, be represented in first-order logic.[3] One of these is:

Some relative of each villager and some relative of each townsman hate each other.

[1] Cf. my "For Every *A* There Is a *B*," *Linguistic Inquiry*, XII (1981): 465-467.
[2] See the entry for 'for' in the Oxford English Dictionary.
[3] "Quantifiers vs. Quantification Theory," *Linguistic Inquiry*, V (1974): 153-177.

There appears to be a consensus regarding this sentence, viz., that if it is O.K., then it can be symbolized in standard first-order logic as follows:

$$\forall x \forall y \exists z \exists w (Vx \,\&\, Ty \rightarrow Rzx \,\&\, Rwy \,\&\, Hzw \,\&\, Hwz \,\&\, z \neq w)$$

I find this sentence marginally acceptable at best and not acceptable if not symbolizable as above.

Jon Barwise has offered "The richer the country, the more powerful is one of its officials" as another example of a sentence that cannot be symbolized in first-order logic.[4] However, since the sentence seems to me, at any rate, to mean "Whenever x is a richer country than y, then x has (at least) one official who is more powerful than *any* official of y," it also seems to me to have a first-order symbolization:

$$\forall x \forall y ([Cx \,\&\, Cy \,\&\, xRy] \rightarrow \exists w [wOx \,\&\, \forall z (zOy \rightarrow wPz)])$$

Are there better examples?

Perhaps the best-known example of a sentence whose quantificational structure cannot be captured by means of first-order logic is the Geach-Kaplan sentence, cited by W. V. Quine in *Methods of Logic*[5] and *The Roots of Reference*[6]:

(A) Some critics admire only one another.

(A) is supposed to mean that there is a collection of critics, each of whose members admires no one not in the collection, and none of whose members admires himself. If the domain of discourse is taken to consist of the critics and Axy to mean "x admires y," then (A) can be symbolized by means of the *second*-order sentence:

$$\text{(B)} \qquad \exists X (\exists x\, Xx \,\&\, \forall x \forall y [Xx \,\&\, Axy \quad x \neq y \,\&\, Xy])$$

And since (B) is not equivalent to any first-order sentence, (A) cannot be correctly symbolized in first-order logic.

The proof, due to David Kaplan, that (B) has no first-order equivalent is simple and exhibits an important technique in showing nonfirstorderizability: Substitute the formula $(x = 0 \vee x = y + 1)$ for Axy in (B), and observe that the result:

$$\text{(C)} \quad \exists X (\exists x\, Xx \,\&\, \forall x \forall y [Xx \,\&\, (x = 0 \vee x = y + 1) \rightarrow x \neq y \,\&\, Xy])$$

[4] "On Branching Quantifiers in English," *Journal of Philosophical Logic*, VIII, 1 (February 1979): 47–80.

[5] 4th ed. (Cambridge, Mass.: Harvard, 1982), p. 293, where "people" is substituted for "critics" in the example.

[6] LaSalle, Ill.: Open Court, 1973; p. 111.

is a sentence that is true in all nonstandard models of arithmetic but false in the standard model.[7]

I must confess to a certain ambivalence regarding the Geach-Kaplan sentence. Although it usually strikes me as a quite acceptable sentence of English, it doesn't invariably do so. (The "only" seems to want to precede the "admires" but the intended meaning of the sentence forces it to stay put.) I find that if the predicates in the example are changed in what one might have supposed to be an inessential way matters are improved slightly:

Some computers communicate only with one another.
Some Bostonians speak only to one another.
Some critics are admired only by one another.

I don't have any idea why replacing the transitive verb 'admires' by a verb or verb phrase taking an accompanying prepositional phrase helps matters, but it does seem to me to do so.

I turn now from this brief survey of known examples of sentences not representable in first-order logic to examination of some other nonfirstorderizable sentences. Like the Geach-Kaplan sentence but unlike the sentences involving 'most', these sentences *look* as if they "ought to be" symbolizable in first-order logic. They contain plural forms such as 'are' and 'them', and it is in large measure because they contain these forms that they cannot be represented in first-order logic.

Consider first the following sentence, which, however, contains no plurals and which can be symbolized in first-order logic:

(D) There is a horse that is faster than Zev and also faster than the sire of any horse that is slower than it.

Quantifying over horses, and using 0, s, $>$, and $<$ for 'Zev', 'the sire of', 'is faster than', and 'is slower than', respectively, we may symbolize (D) in first-order logic:

(E) $$\exists x(x > 0 \mathbin{\&} \forall y[y < x \rightarrow x > s(y)])$$

[7] To see that (C) is true in any nonstandard model, take as X the set of all nonstandard elements of the model. X is nonempty, does not contain 0, hence contains only successors, and contains the immediate predecessor of any of its members. To see that it is false in the standard model, suppose that there is some suitable set X of natural numbers. X must be nonempty: if its least member x is 0, let $y = 0$; otherwise $x = y + 1$ for some y. Since x is least, y is not in X, and 'Xy' is false. The nonfirstorderizability of "For every A there is a B" can be established in a similar way: Select variables x and y not found in any presumed first-order equivalent, substitute $[(1) < x + 5 \mathbin{\&} \sim\exists y 3 \cdot y = (1)]$ for $A(1)$, substitute $[(1) < x + 5 \mathbin{\&} \exists y 3 \cdot y = (1)]$ for $B(1)$, and existentially quantify the result with respect to x; the result would be true in all nonstandard models but false in the standard model.

Sentence (F), however, cannot be symbolized in first-order logic:

(F) There are some horses that are faster than Zev and also faster than the sire of any horse that is slower than them.

(F) differs from (D) only in that some occurrences in (D) of the words 'is', 'a', 'horse', and 'it' have been replaced by occurrences of their plural forms 'are', 'some', 'horses', and 'them'. The content of (F) is given slightly more explicitly in:

(G) There are some horses that are all faster than Zev and also faster than the sire of any horse that is slower than all of them.

I take it that (F) and its variant (G) can be paraphrased: there is a nonempty collection (class, totality) X of horses, such that all members of X are faster than Zev and such that, whenever any horse is slower than all members of X, then all members of X are faster than the sire of that horse.[8] (F) and (G) can be symbolized by means of the second-order sentence (domain and denotations as above):

(H) $\exists X(\exists x\, Xx \;\&\; \forall x(Xx \rightarrow x > 0)$
$\qquad \& \; \forall y[\forall x(Xx \rightarrow y < x) \rightarrow \forall x(Xx \rightarrow x > s(y))])$

(H) is equivalent to no first-order sentence; for it is false in the standard model of arithmetic (under the obvious reinterpretation) but true in any nonstandard model, since the set of nonstandard elements of the model will always be a suitable value for X. Thus (F) cannot be symbolized in first-order logic.[9]

(F) is not an especially pretty sentence. It is hard to understand, awkward, and contrived. But ugly or not, it is a perfectly grammatical sentence of English, which has, as far as I can see, the meaning given above and no other. Moreover, such faults as it has appear to be fully shared by (D).

Another example, shorter and perhaps more intelligible:

(I) There are some gunslingers each of whom has shot the right foot of at least one of the others.

[8] Zev won the Kentucky Derby in 1923.

[9] Cf. my "Nonfirstorderizability Again," to appear in *Linguistic Inquiry*, xv, 2 (1984). In an important unpublished manuscript entitled "Plural Quantification," Lauri Carlson has given "If some numbers all are natural numbers, one of them is the smallest of them," as an example of a sentence that cannot be symbolized in the first-order predicate calculus. I have heard it claimed that this is not a proper sentence of English. Perhaps it is not, but "If there are some numbers all of which are natural numbers, then there is one of them that is smaller than all the others," surely is. I am grateful to Irene Heim for calling this reference to my attention.

(I) may be rendered in second-order logic:

(J) $$\exists X(\exists x\, Xx \mathbin{\&} \forall x[Xx \rightarrow \exists y(Xy \mathbin{\&} y \neq x \mathbin{\&} Bxy)])$$

(Here we quantify over gunslingers and use B for "has shot the right foot of.") By substituting $x = y + 1$ for Bxy, we may easily see that (J) is equivalent to no first-order sentence. (Alternatively, we may note that if we negate (J), substitute $y \leq x$ for Bxy, and make some elementary transformations, we obtain:

$$\forall X(\exists x\, Xx \rightarrow \exists x[Xx \mathbin{\&} \forall y(Xy \mathbin{\&} y \leq x \rightarrow y = x)])$$

a formula that expresses the least-number principle, which is one version of the principle of mathematical induction.)

When used as a demonstrative pronoun, 'that' is marked for number, as singular, but when used as a relative pronoun, as in (F), it is unmarked for number, i.e., can be used in either the singular or plural. 'Who', 'whom', and 'whose', however, are unmarked for number when used either as relative or as interrogative pronouns. 'Which' is also unmarked for number as a relative pronoun, but 'which ones', when it can be used, is strongly preferred to 'which' as an interrogative plural form; it may well be that interrogative 'which', like demonstrative 'that', is marked as singular.

It is the plural forms in (F) and (I), as well as the unmarkedness of 'that' and 'whom', that are responsible for the nonfirstorderizability of these sentences. And by taking a cue from the well-known second-order definitions of "x is a standard natural number" and "x is an ancestor of y," we can use plurals to define these notions in English (in terms of "zero" and "successor of" and in terms of "parent of," respectively):

(K) If there are some numbers of which the successor of any one of them is also one, then if zero is one of them, x is one of them.

(L) If there are some persons of whom each parent of any one of them is also one, then if each parent of y is one of them, x is one of them; and someone is a parent of y.

There are some comments on (K) and (L) to be made: (a) 'which' and 'whom' are used in these sentences as we have noticed they can be used, in the plural. (b) Instead of saying "of which the successor of any one of them is also one," one could as well say "of which the successor of any one is also one of them": at least one "them" is needed to cross-refer to the "witnessing" values of 'which': this 'them' is sometimes called a *resumptive* pronoun, and appears to be needed to capture the force of $\forall y(Xy \rightarrow Xs(y))$, with its two occurrences of X. (c) Like (F) and (I), (K) and (L) cannot be given correct first-order symbolizations, and thus the following (valid) inference cannot be represented in first-order logic:

> If there are some persons of whom each parent of any one of them is also one, then if each parent of Yolanda is one of them, Xavier is one of them; and someone is a parent of Yolanda.
> Every parent of someone red is blue.
> Every parent of someone blue is red.
> Yolanda is blue.
> Therefore, Xavier is either red or blue.

(To see that this is a valid inference, consider the persons who are either red or blue. By the second and third premises, every parent of any one of these persons is also one of them; and since Yolanda is blue, each of her parents is red, hence red or blue, and hence one of these persons. Thus Xavier is also one of them and thus either red or blue.) (d) The 'there are's in the antecedents of course express universal quantification, as does the 'there is' in "If there is a logician present, he should leave." (e) Like (F), (K) and (L) are somewhat ungainly, in part because of the resumptive 'them' they contain, but principally because of the complexity of the thoughts they express. However, they seem to be perfectly acceptable vehicles for the expression of those very thoughts. And although they are indeed contrived—they have been contrived *to take advantage of referential devices that are available in English*—the fact that they are so hardly begins to bear on the question whether they are ungrammatical, unintelligible, or in some other way unacceptable.

The suggestion that it is the complexity of the thoughts expressed in (K) and (L) that is responsible for their ungainliness rather than the presence of any construction not properly a part of English draws support from the ease and naturalness with which "x is identical to y" may be defined in the same style: if there are some things of which x is one, then y is one of them too. (Or: it is not the case that there are some things of which x is one, but of which y is not one.)

Another example, of a different sort, is:

(M) Each of the numbers in the sequence 1,2,4,8, . . . is greater than the sum of all the numbers in the sequence that precede it.

(M) states something true, which, using a mixture of logical and arithmetical notation, we can express as follows:

$$\text{(N)}\quad \forall x \forall y (Px \,\&\, y = \Sigma\{z : Pz \,\&\, x > z\} \rightarrow x > y)$$

In (N), Σ is a sign for a function from sets of objects in a domain to objects in that domain and attaches to a variable and a formula to form a term in which that variable is bound. Signs for such functions are simply not part of the primitive vocabulary of first-order logic, although on occasion mention of functions of this type can be paraphrased away (e.g. "the least of the numbers z such that . . . z . . ."). No one function sign of the ordinary sort can do full justice to "the sum of the numbers z such that . . . z . . . ," as can be seen by considering:

(O) Although every power of 2 is 1 greater than the sum of all the powers of 2 that are smaller than it, not every power of 3 is 1 greater than the sum of all the powers of 3 that are smaller than it.

We certainly cannot symbolize (O) as:

$$\forall x \forall y (Px \ \&\ y = f(x) \rightarrow x = y + 1) \ \&\ \sim\forall x \forall y (Qx \ \&\ y = f(x) \rightarrow x = y + 1)$$

and were we to try to improve matters by changing the second occurrence of f to an occurrence of (say) g, we should fail to depict the recurrence of the semantic primitive 'the sum of . . .' in the second conjunct of (O). Nor could any ordinary function sign express the dependencies that may obtain between predicates contained in '. . . z . . .' and those found in the surrounding context.

A short and sweet example of the same type is:

No number is the sum of all numbers.

The last example for the moment of a sentence whose meanings cannot all be captured in first-order logic is one that is again found in Quine's *Methods of Logic*—but not, this time, in the final part of the book, "Glimpses Beyond." It is the sentence (P):

(P) Some of Fiorecchio's men entered the building unaccompanied by anyone else.

On Quine's analysis of this sentence, it can be represented as $\exists x(Fx \ \&\ Ex \ \&\ \forall y[Axy \rightarrow Fy])$, where Fx, Ex, and Axy mean "x was one of Fiorecchio's men," "x entered the bulding," and "x was accompanied by y."[10] Quine states that "x was unaccompanied by anyone else" clearly has the intended meaning "Anyone accompanying x was one of Fiorecchio's men."

Quine's is certainly one reading this sentence bears: there are some Fiorecchians each of whom entered the building unaccom-

[10] Page, 197. Quine uses K, F, and H instead of F, E, and A, respectively.

panied by anyone who wasn't a Fiorecchian. But since (P) appears, at times, to mean something like:

> There were some men, see.
> They were all Fiorecchio's men.
> They entered the building.
> And they weren't accompanied by anyone else.

it can also be understood to mean: there are some Fiorecchians each of whom entered the building unaccompanied by anyone who wasn't one of *them*. On this stronger reading, there is no asymmetry between the predicates "x was one of Fiorecchio's men" and "x entered the building," 'else' means "not one of them," and the whole can be symbolized by:

$$\exists X(\exists x\, Xx \,\&\, \forall x(Xx \rightarrow Fx) \,\&\, \forall x(Xx \rightarrow Ex) \,\&\, \forall x \forall y(Xx \,\&\, Axy \rightarrow Xy))$$

whose nonfirstorderizability can be seen in the usual way, by substituting $x > 0$ for both Fx and Ex and $x = y + 1$ for Axy.

It is because of these examples that I think that the question whether the first-order predicate calculus with identity adequately represents quantification, generalization, and cross reference in natural language ought to be regarded as a question that hasn't yet been settled.

Changing the subject somewhat, I now want to look at a number of sentences whose most *natural* representations are given by second-order formulas, but second-order formulas that turn out to be equivalent to first-order formulas.

The sentence:

(Q) There are some monuments in Italy of which no one tourist has seen all.

might appear to require a second-order formula for its correct symbolization, e.g.,

(R) $\exists X(\exists x\, Xx \,\&\, \forall x[Xx \rightarrow Mx] \,\&\, \sim\exists y[Ty \,\&\, \forall x(Xx \rightarrow Syx)])$

Of course, (Q) can be paraphrased:

(S) No tourist has seen all the monuments in Italy.

and this can be symbolized in first-order logic as:

(T) $\exists x Mx \,\&\, \sim\exists y[Ty \,\&\, \forall x(Mx \rightarrow Syx)]$

which is equivalent to (R).[11] But just as $\sim\sim p$ can sometimes be a

[11] It take it that since (S) implies that there are some monuments in Italy, but does not imply that there are tourists, the conjunct $\exists x Mx$ is indispensable.

better symbolization than p of "It's not the case that John didn't go," e.g., if p were used to symbolize "John went," so (R) captures more of the quantificational structure of (Q) than does the equivalent (T). (Q) might appear to say that there is a (nonempty) collection of monuments in Italy and no tourist has seen every member of this collection; (S) doesn't begin to hint at collections of monuments. Nevertheless, (Q) and (S) say the same thing, if any two sentences do, and (R) and (T) are, predictably enough, equivalent.

Another example of the same "collapsing" phenomenon:

(U) Mozart composed a number of works, and every tolerable opera with an Italian libretto is one of them.

has the second-order symbolization:

(V) $$\exists X(\exists x Xx \,\&\, \forall x(Xx \rightarrow Mx) \,\&\, \forall x(Tx \rightarrow Xx))$$

But as (U) says what (W) says:

(W) Mozart composed a number of works, and every tolerable opera with an Italian libretto is a work that Mozart composed.

so (V) is equivalent to the first order

(X) $$\exists x Mx \,\&\, \forall x(Tx \rightarrow Mx)$$

The construction 'Every . . . is one of them' bears watching; suffice it for now to observe that it is a perfectly ordinary English phrase.

Collapses can also occur unexpectedly. (Through a publisher's error) the sentence:

(Y) Some critics admire one another and no one else.

meaning (approximately), "There is a collection of critics, each of whom admires all and only the *other* members of the collection," and possessing the second-order symbolization:

(Z) $$\exists X(\exists x \exists y[Xx \,\&\, Xy \,\&\, x \neq y] \\ \&\, \forall x[Xx \rightarrow \forall y(Axy \leftrightarrow \{Xy \,\&\, y \neq x\})])$$

was claimed in the first American printing of the third edition of *Methods of Logic* to be a sentence incapable of first-order representation.[12] But although (Z) might appear to be susceptible to the same kind of treatment given out above, it was in fact observed by Kaplan to be equivalent to the first-order formula:

(a) $$\exists z(\exists y Azy \,\&\, \forall x[(z = x \vee Azx) \\ \rightarrow \forall y(Axy \leftrightarrow \{(z = y \vee Azy) \,\&\, y \neq x\})])$$

[12] *Methods of Logic*, 3d ed. (New York: Holt, Rinehart & Winston, 1972), p. 238/9.

Consider now sentence (b):

(b) There are some sets that are such that no one of them is a member of itself and also such that every set that is not a member of itself is one of them. (Alternatively: There are some sets, no one of which is a member of itself, and of which every set that is not a member of itself is one.)

By quantifying over sets and abbreviating "is a member of" by ϵ, we may use a second-order formula to symbolize (b):

(c) $$\exists X(\exists x\, Xx \;\&\; \forall x[Xx \rightarrow \sim x\epsilon x] \;\&\; \forall x[\sim x\epsilon x \rightarrow Xx])$$

(c) is obviously equivalent to (d):

(d) $$\exists X(\exists x\, Xx \;\&\; \forall x[Xx \leftrightarrow \sim x\epsilon x])$$

Let us notice that (d) immediately implies $\exists x \sim x\epsilon x$. Conversely, if $\exists x \sim x\epsilon x$ holds, then there is at least one set in the totality X of sets that are not members of themselves, and X witnesses the truth of (d). Thus (d) turns out to be equivalent to $\exists x \sim x\epsilon x$, the symbolization of an obvious truth concerning sets.

(The worry over Russell's paradox which the reader may be experiencing at this point may be dispelled by the observation that logical equivalence is a model-theoretic notion, the "sets" just referred to may be taken to be elements of the domain of an arbitrary model, and the "totalities," subsets of the domain of the model.)

In view of the near-vacuity of (b) and the fact that instances of the second-order comprehension schema $\exists X\, \forall x[Xx \leftrightarrow A(x)]$, including (e):

(e) $$\exists X\, \forall x[Xx \leftrightarrow \sim x\epsilon x]$$

are logically valid under the standard semantics for second-order logic, the collapse of (d) is not at all surprising. The rendering (d) of (b) is considerably more faithful to the semantic structure of (b) than is $\exists x \sim x\epsilon x$, however, and (b) is more nearly synonymous with (d) than with $\exists x \sim x\epsilon x$.

But can we use (c) or (d) to represent (b) at all? May we use second-order formulas like (c), (d), or (e) to make assertions about *all sets*?

Let's consider (e), which is slightly simpler than (c) or (d). (e) would appear to say that there is a totality or collection X containing all and only those sets x which are not members of themselves. Are we not here on the brink of a well-known abyss? Does not acceptance of the valid (e), understood as quantifying over all sets (with ε taken to have its usual meaning), commit us to the existence of a set whose members are all and only those sets which are not members of themselves?

There are a number of ways out of this difficulty. One way, which I no longer favor, is to regard it as illegitimate to use a second-order formula when the objects over which the individual variables in the formula range do not form a set (just as it is illegitimate to use a first-order formula when there are *no* objects over which they range).[13] This stipulation keeps all instances of the comprehension principle as logical truths; it also enables one always to read the formula Xx as meaning that x is a *member* of the *set* X.

The principal drawback of this way out is that there are certain assertions about sets that we wish to make, which certainly cannot be made by means of a first-order formula—perhaps to claim that there is a "totality" or "collection" containing all and only the sets that do not contain themselves is to attempt to make one of these assertions—but which, it appears, could be expressed by means of a second-order formula if only it were permissible so to express them. To declare it illegitimate to use second-order formulas in discourse about all sets deprives second-order logic of its utility in an area in which it might have been expected to be of considerable value.

For example, the principle of set-theoretic induction and the separation (Aussonderung) principle virtually cry out for second-order formulation, as:

$$\text{(f)} \qquad \forall X(\exists x\, Xx \rightarrow \exists x[Xx \,\&\, \forall y(y \epsilon x \rightarrow {\sim}Xy)])$$

and

$$\text{(g)} \qquad \forall X\, \forall z\, \exists y\, \forall x(x \epsilon y \leftrightarrow [x \epsilon z \,\&\, Xx])$$

respectively. It is, I think, clear that our decision to rest content with a set theory formulated in the first-order predicate calculus with identity, in which (f) and (g) are not even well-formed, must be regarded as a compromise, as falling short of saying all that we might hope to say. Whatever our reasons for adopting Zermelo-Fraenkel set theory in its usual formulation may be, we accept this theory because we accept a stronger theory consisting of a *finite* number of principles, among them some for whose complete expression second-order formulas are required.[14] We ought to be able to formulate a theory that reflects our beliefs.

[13] I took this view in "On Second-order Logic," this JOURNAL, LXXII, 16 (Sept. 18, 1975): 509-527.

[14] Cf. the remarks about "full expression" and "part of the content" of various notions in my "The Iterative Conception of Set," this JOURNAL, LXVIII, 8 (April 22, 1971): 215-231.

We of course also wish to *maintain* such second-claims as are made by e.g., $\exists X \forall x[Xx \leftrightarrow \sim x\epsilon x]$; if we are to utilize second-order logic in discourse about all sets, these comprehension principles must remain among the asserted statements. Nor do we want to take the second-order variables as ranging over some set-like objects, sometimes called "classes," which have members, but are not themselves members of other sets, supposedly because they are "too big" to be sets. Set theory is supposed to be a theory about *all* set-like objects.

How then can we legitimately maintain that such (closed) formulas as $\exists X \forall x[Xx \leftrightarrow \sim x\epsilon x]$, (f), and (g) express truths, without introducing classes (set-like non-sets) into set theory and without assuming that the individual variables do not in fact range over all the sets there really are?

There is a simple answer. Abandon, if one ever had it, the idea that use of plural forms must always be understood to commit one to the existence of sets (or "classes," "collections," or "totalities") of those things to which the corresponding singular forms apply. The idea is untenable in general in any event: There are some sets of which every set that is not a member of itself is one, but there is no set of which every set that is not a member of itself is a member, as the reader, understanding English and knowing some set theory, is doubtless prepared to agree. Then, using the plural forms that are available in one's mother tongue, translate the formulas into that tongue and see that the resulting English (or whatever) sentences express true statements. The sentences that arise in this way will lack the trenchancy of memorable aphorisms, but they will be proper sentences of English which, with a modicum of difficulty, can be understood and seen to say something true.

Applying this suggestion to:

(h) $\sim\exists X(\exists x Xx \ \&\ \forall x[Xx \rightarrow (x\epsilon x \vee \exists y[y\epsilon x \ \&\ Xy \ \&\ y \neq x])])$

which is equivalent to (f), we might obtain:

(i) It is not the case that there are some sets each of which either contains itself or contains at least one of the others.

From Aussonderung we might perhaps get:

(j) It is not the case that there are some sets that are such that it is not the case that for any set z there is a set y such that for any set x, x is a member of y if and only if x is a member of z and also one of them.

or, far more perspicuously,

(k) $\sim$ there are some sets such that
$\sim \forall z \exists y \forall x[x \epsilon y \leftrightarrow (x \epsilon z \;\&\; x$ is one of them)]

(k) is of course neither an English sentence nor a wff of any reputable formalism—for that matter neither is (j), which contains the (non-English) variables x, y, and z—but is readily understood by anyone who understands both English and the first-order language of set theory. It would be somewhat laborious to produce a fully Englished version of (g), but the labor involved would be mainly due to the sequence $\forall x \exists y \forall x$ of *first-order* quantifiers that (g) contains. (j) and (k) are actually not quite right; properly they have the meaning:

(l) $$\sim \exists X(\exists x\, Xx \;\&\; \sim \forall z \exists y \forall x[x \epsilon y \leftrightarrow (x \epsilon z \;\&\; Xx)])$$

whereas the full Aussonderung principle omits the nonemptiness condition $\exists x\, Xx$; to get the full content in English of Aussonderung, however, we need only conjoin "and there is a set with no members" to (j) and $\exists y \forall x \sim x \epsilon y$ to (k). This observation calls to our attention two small matters connected with plurals which must be taken up sooner or later.

Suppose that there is exactly one Cheerio in the bowl before me. Is it true to say that there are some Cheerios in the bowl? My view is no, not really, I guess not, but say what you like, it doesn't matter very much. Throughout this paper I have made the customary logician's assumption, which eliminates needless verbiage, that the use of plural forms does not commit one to the existence of two or more things of the kind in question.

On the side of literalness, however, I have assumed that use of such phrases as "some gunslingers" in "There are some gunslingers each of whom has either shot his own right foot or shot the right foot of at least one of the others" does commit one to—as one might say—a *nonempty* class of gunslingers, but not to a class containing two or more of them. Thus I suppose the sentence to be true in case there is exactly one gunslinger, who has shot his own right foot, but to be false if there are aren't any gunslingers. It is this second assumption that is responsible for the ubiquitous $\exists x\, Xx$ in the formulas above.

Translation will be difficult from any logical formalism into a language such as English, which lacks a large set of devices for expressing cross reference. And since plural pronouns like 'them', although sometimes used as English analogues of second-order variables, much more frequently do the work of individual variables, translation from a second-order formalism containing infinitely many variables of both sorts into idiomatic, flowing, and easily understood English will be impossible nearly all of the time. My present point is that, in the cases of interest to us, the things we would like to say can be said, if not with Austinian or Austenian grace.

It is, moreover, clear that if English were augmented with various subscripted pronouns, such as 'it$_x$', 'that$_x$', 'it$_y$', . . . , 'them$_X$', 'that$_X$', 'them$_Y$', . . . , then any second-order formula[15] whose individual variables are understood to range over all sets could be translated into the augmented language, as follows: Translate Vv as 'it$_v$ is one of them$_V$', $v \varepsilon v'$ as 'it$_v$ is a member of it$_{v'}$', $v = v'$ as 'it$_v$ is identical with it$_{v'}$', & as 'and', ~ as 'not', and, where F^* is the translation of F, translate $\exists vF$ as 'there is a set that$_v$ is such that F^*'.

The clause for formulas $\exists VF$ is not quite so straightforward, because of the difficulty about nonemptiness mentioned above. It runs as follows: Let F^* be the translation of F, and let F^{**} be the translation of the result of substituting an occurrence of $\sim v = v$ for each occurrence of Vv in F. Then translate $\exists VF$ as 'either there are some sets that$_V$ are such that F^*, or F^{**}'.

For example, $(Xx \leftrightarrow \sim x \epsilon x)$ comes out as "It$_x$ is one of them$_X$ iff it$_x$ is not a member of itself"; $\forall x(Xx \leftrightarrow \sim x \epsilon x)$, as "Every set is such that it is one of them$_X$ iff it is not a member of itself"; and $\exists X \forall x(Xx \leftrightarrow \sim x \epsilon x)$, as "Either there are some sets that are such that every set is one of them iff it is not a member of itself or every set is a member of itself." (We have, of course, improved the translations as we went along.)

I want to emphasize that the addition to English of operators 'it$_{()}$', 'that$_{()}$', 'them$_{()}$', etc. or variables 'x', 'X', 'y', etc. is not contemplated here. The 'x' of 'it$_x$' is not a variable but an index, analogous to 'latter' in 'the latter', or 'seventeen' in 'party of the seventeenth part'; 'X' and 'x' in 'them$_X$' and 'it$_x$' no more have *ranges* or *domains* that does '17' in 'x_{17}'. We could just as well have translated the language of second-order set theory into an English augmented with pronouns such as 'it$_{17}$', 'them$_{1879}$', etc. or an elaboration of the "former"/"latter" usage. Note also that such augmentation will be needed for the translation into English of the language of *first*-order set theory as well.

[15] We assume that no quantifier in any formula occurs vacuously or in the scope of another quantifier with the same variable; every formula is equivalent to some formula satisfying this condition.

Charles Parsons has pointed out to me that although second-order existential quantifiers can be rendered in the manner we have described, it is curious that there appears to be no nonartificial way to translate second-order *universal* quantifiers, that the translation of $\forall X$ must be given indirectly, via its equivalence with $\sim\exists X\sim$. Because our translation "manual" relies so heavily on the phrases 'there is a [singular count noun] that is such that . . . it . . .' and 'there are some [plural count noun] that are such that . . . they . . .', the logical grammar of the construction these phrases exemplify is worth looking at.

Of course, in ordinary speech, the construction 'that is/are such that . . . it/they . . .' is almost certain to be eliminable: the content of a sentence containing it can nearly always be conveyed in a much shorter sentence. But the difference between the two 'that's bears notice. The second one, following 'such', is a 'that' like the one found in oblique contexts and may be—as Donald Davidson has suggested that the 'that' of indirect discourse is—a kind of demonstrative, used on an occasion to point to a subsequent utterance of an (open) sentence; the first 'that', following the count noun and more frequently elided than the second, is no demonstrative, but a relative pronoun used to bind the 'it' or 'they' in the open sentence after 'such that'. Thus the first but not the second 'that' works rather like the variable immediately following an $\exists$, binding occurrences of that same variable in a subsequent open formula. Whether the preceding count noun is singular or plural appears to make no difference to the quantificational role of the first 'that'; as we have observed, 'that' is not marked for number and can serve to bind either 'it' or 'they'.

Whether any such second-order formula of the sort we have been considering can be translated into intelligible *un*augmented English is not an interesting question, and I shall leave it unanswered. Since English augmented in the manner I have described is intelligible to any native speaker who understands the term of art 'party of the seventeenth part', I shall assume that devices like 'it$_x$' and 'them$_X$' are available in the language we use.

I take it, then, that there is a coherent and intelligible way of interpreting such second-order formulas as (e), (f), and (g) even when the first-order variables in these formulas are construed as ranging over all the sets or set-like objects there are. The interpretation is given by translating them into the language we speak; the translations of (e), (f), and (g) are sentences we understand; and we can see that they express statements that we regard as true: after all, we do think it false that there are some sets each of which either contains itself or contains one of the others, and, once we cut through the verbiage, we do find it trivial that there are some sets none of which is a member of itself and of which each set that is not a member of itself is one. It cannot seriously be maintained that we do not *understand* these statements (unless of course we really *don't* understand them, as we wouldn't if, e.g., we knew nothing at all about set theory) or that any lack of clarity that attaches to them has anything to do with the plural forms found in the sentences expressing them. The language in which we think and speak provides the constructions and turns of phrase by means of which the meanings of these formulas may be explained in a completely intelligible way.

It may be suggested that sentences like (i) are intelligible, but only because we antecedently understand statements about collections, totalities, or sets, and that these sentences are to be analyzed as claims about the existence of certain collections, etc. Thus "There are some gunslingers . . ." is to be analyzed as the claim that there is a collection of gunslingers. . . .[16] The suggestion may arise from the thought that any precise and adequate semantics for natural language must be interpretable in set theory (with individuals). How else, one may wonder, is one to give an account of the semantics of plurals?

One should not confuse the question whether certain sentences of our language containing plurals are intelligible with the question whether one can give a semantic theory for those sentences. In view of the work of Tarski, it should not automatically be expected that we can give an adequate semantics for English—whatever that might be—in English. Nothing whatever about the intelligibility of those sentences would follow from the fact that a systematic semantics for them cannot be given in set theory. After all, the semantics of the language of ZF itself cannot be given in ZF.

In any event, as we have noticed, there are certain sentences that cannot be analyzed as expressing statements about collections in the manner suggested, e.g., "There are some sets that are self-identical, and every set that is not a member of itself is one of them."

[16] In a similar vein, Lauri Carlson writes, "I take such observations as a sufficient motivation for construing *all* plural quantifier phrases as quantifiers over arbitrary *sets* [Italics Carlson's] of those objects which form the range of the corresponding singular quantifier phrases." His "Plural Quantifiers and Informational Independence," *Acta Philosophica Fennica*, xxxv (1982): 163–174, is a recent interesting article in which this claim is made once again. He is by no means the sole linguist with this belief. Carlson does not face the question of what is to be done when the corresponding singular quantifier phrase is 'some set'.

That sentence says something trivially true; but the sentence "There is a collection of sets that are self-identical, and every set that is not a member of itself is a member of this collection," which is supposed to make its meaning explicit, says something false.

I want now to consider the claim that a sentence of English like "There are some sets of which every set that is not a member of itself is one" is actually false, on the ground that this sentence *does* entail the existence of an overly large set, one that contains all sets that are not members of themselves.

The claim that this sentence entails the existence of this large set strikes me as most implausible: there may be a set containing all trucks, but that there is certainly doesn't seem to *follow* from the truth of "There are some trucks of which every truck is one." Moreover, and more importantly, the claim conflicts with a strong intuition, which I for one am loath to abandon, about the meaning of English sentences of the form "There are some *A*s of which every *B* is one," viz. that any sentence of this form means the same thing as the corresponding sentence of the form "There are some *A*s and every *B* is an *A*." If so, the sentence of the previous paragraph is simply synonymous with the trivial truth "There are some sets and every set that is not a member of itself is a set," and therefore does not entail the existence of an overly large set.

Two worries of a different kind are that the construction "there are some [plural count noun] that are such that . . . they . . .' is unintelligible if the individuals in question do not form a "surveyable" set and that our understanding of this construction does not justify acceptance of full comprehension. I cannot deal with these worries here; I shall only remark that it seems likely that not much of ordinary, first-order, set theory would survive should either worry prove correct.

We have now arrived at the following view: Second-order formulas in which the individual variables are taken as ranging over all sets can be intelligibly interpreted by means of constructions available to us in a language we already understand; these constructions do not themselves need to be understood as quantifying over any sort of "big" objects which have members and which "would be" sets "but for" their size. There can thus be no objection on the score of unintelligibility or of the introduction of unwanted objects to our regarding ZF as more suitably formulated as a finitely axiomatized second-order theory than as an infinitely axiomatized first-order theory, whose axioms are the instances of a finite number of schemata, as is usual. (Of course, in the presence of the usual other first-order axioms of ZF, i.e., the axioms of extensionality, foundation, pairing, power set, union, infinity, and choice, only the one second-order axiom, Replacement:

$$\forall X(\forall x \forall y \forall z[X\langle x,y\rangle \,\&\, X\langle x,z\rangle \rightarrow y = z] \rightarrow \forall u \exists v \forall y[y\epsilon v \leftrightarrow \exists x(x\epsilon u \,\&\, X\langle x,y\rangle)])$$

would be needed.) The great virtue of such a second-order formulation of ZF is that it would permit us to express as single sentences and take as axioms of the theory certain general principles that we actually believe. The underlying logic of such a formulation would be any standard axiomatic system of second-order logic, e.g., the system indicated, if not given with perfect precision, in Frege's *Begriffsschrift*.[17] The logic would deliver the comprehension principles $\exists X \forall x[Xx \leftrightarrow A(x)]$ (which are needed for the derivation of the infinitely many axioms of the first-order version of ZF from the finitely many second-order axioms) either through explicit postulation of the comprehension schema, as in Joel Robbin's *Mathematical Logic*,[18] or via a rule of substitution, like the rule given in chapter 5 of Alonzo Church's *Introduction*[19] or the one implicit in the *Begriffsschrift*. The interpretation of this version of ZF would be given in a manner similar to that in which the interpretation of the usual formulation of ZF is given, by translation into English in the manner previously described.

Entities are not to be multiplied beyond necessity. One might doubt, for example, that there is such a thing as the set of Cheerios in the (other) bowl on the table. There are, of course, quite a lot of Cheerios in that bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? And what about the $>10^{60}$ subsets of that set? (And don't forget the sets of sets of Cheerios in the bowl.) It is haywire to think that when you have some Cheerios, you are eating a *set*—what you're doing is: eating THE CHEERIOS. Maybe there are some reasons for thinking there is such a set—there are, after all, $>10^{60}$ ways to divide the Cheerios into two portions—but it doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize

[17] Gottlob Frege (Hildesheim: Georg Olms Verlagsbuchhandlung, 1964).

[18] *Mathematical Logic: A First Course* (New York: W. A. Benjamin, 1969). Section 56 of Robbin's book contains a presentation of the version of set theory here advocated. It is noted there that this theory is "essentially the same as" Morse-Kelley set theory (MK), but the difficulties of interpretation faced either by MK or by a set theory in the ZF family for which the underlying logic is (axiomatic) second-order logic are not discussed.

[19] *Introduction to Mathematical Logic* (Princeton, N.J.: University Press, 1956).

about the semantics of plurals would have it, there is also a set of them all.

The lesson to be drawn from the foregoing reflections on plurals and second-order logic is that neither the use of plurals nor the employment of second-order logic commits us to the existence of extra items beyond those to which we are already committed. We need not construe second-order quantifiers as ranging over anything other than the objects over which our first-order quantifiers range, and, in the absence of other reasons for thinking so, we need not think that there are collections of (say) Cheerios, in addition to the Cheerios. Ontological commitment is carried by our *first*-order quantifiers; a second-order quantifier needn't be taken to be a kind of first-order quantifier in disguise, having items of a special kind, collections, in its range. It is not as though there were two sorts of things in the world, individuals, and collections of them, which our first- and second-order variables, respectively, range over and which our singular and plural forms, respectively, denote. There are, rather, two (at least) different ways of referring to the same things, among which there may well be many, many collections.

Leibniz once said, "Whatever is, is one."

Russell replied, "And whatever are, are many."[20]

GEORGE BOOLOS

Massachusetts Institute of Technology

BOOK REVIEWS

The Nature of Mathematical Knowledge. PHILIP KITCHER. New York: Oxford University Press, 1983. 287 p. $25.00.

Kitcher has given us two books, albeit unified by a common point of view: a historically oriented philosophy of mathematics, and a philosophically oriented history of mathematics.* The history part takes up the last four of the ten chapters of *The Nature of Ma-*

[20] Bertrand Russell, *The Principles of Mathematics*, 2d ed. (London: Allen & Unwin, 1937), p. 132.

* These reflections were greatly clarified by interchanges with Charles Parsons and the other members of his Fall, 1983, seminar at Columbia University, which I attended, and at which Kitcher's book was discussed. What I say here, however, does not necessarily reflect the others' reactions to the book. Conversations with Paul Benacerraf, Michael Resnik, and David Shatz also improved the content and the style of this review.

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