Structuralism and the identity of indiscernibles

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1. Mathematical structuralism and the Burgess-Keränen objection

According to mathematical structuralism, mathematics is the science of pattern and structure.[1] A structure, as the notion is understood in contemporary mathematics, is a set (or possibly a proper class) with distin-

[1] See Hellman 1989, Shapiro 1997 and Resnik 1997 for defences of versions of structuralism.

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guished relations (which may be operations). Examples are orderings, rings, groups, fields, lattices, Boolean algebras, trees, etc.

To introduce a central motivation for structuralism, consider the representation of the system **N** of natural numbers within set theory. Suppose we define 0 as Ø. Then we may define the *successor* of x to be $x \cup \{x\}$ or to be $\{x\}$. If we then take the smallest set containing 0 and closed under successor, then both definitions – and indeed countless others – yield isomorphic ω-sequences. But none of these representations seems any more basic or privileged than any other. Questions of the form 'is 2 an *element* of 4?' strike us as pseudo-questions. The conclusion is that whatever natural numbers are, they cannot *be* sets. This line of argument, with structuralism as one resolution, was developed by Paul Benacerraf (1965). The argument concludes that if there is such an entity as the natural number structure **N**, it must be thought of as a structure in some more *abstract* sense, where isomorphic 'systems' count as 'instances' of the same structure. And the conclusion may be generalized to a variety of mathematical structures (e.g. groups, rings, fields, manifolds, etc.).

Stewart Shapiro (1997: 89) has introduced the phrase '*ante rem* structure' to refer to such abstract structures. Shapiro likens such abstract structures to universals or types, while the 'systems' that exemplify them are likened to instances or tokens. While the domain of a system may consist of ordinary objects (in some sense), the domain of an *ante rem* structure comprises 'positions'.[2] These positions are understood to be individuated solely by *their intra-structural relations to each other*: 'there is no more to the individual numbers "in themselves" than the relations they bear to each another' (Shapiro 1997: 73).

A recent objection to mathematical structuralism has been given by John Burgess (1999) and Jukka Keränen (2001). The objection concerns the conception of identity for the 'positions' of *ante rem* structures. If positions are to be individuated by the 'relations they bear to each other', then it seems plausible that the identity of a position should be fixed by its *structural role*. If this thought is right, then *structurally indiscernible positions should be identical*. This corresponds to the following indiscernibility principle:

(I) For any structure **M**, if $a, b \in \mathrm{dom}(\mathbf{M})$ are structurally indiscernible, then $a = b$.

[2] One might argue that a specific abstract structure (e.g. *the* group SU(3), etc.) should be regarded as the *isomorphism type* of all of its instances. But does an isomorphism type have a *domain*? John Burgess notes (1999: 287) that 'an isomorphism type is no more a special kind of system than a direction is a special kind of line'. But while mathematicians do not think of directions as lines, they *do* treat each group, field, etc., as having a specific domain, and distinguished relations and operations.

Intuitively, elements of a structure are structurally indiscernible if some symmetry transformation relates them. The symmetries of a structure are called its automorphisms.[3] The definition we obtain is this: $a, b \in \text{dom}(\mathbf{M})$ are *structurally indiscernible* just in case there is an automorphism π of $\mathbf{M}$ such that $b = \pi(a)$.[4] In geometrical parlance, we say that b lies in the *orbit* of a. The orbits form a partition of the domain into equivalence classes, and structurally indiscernible elements are those which lie in the same orbit.

Consider the complex field $\mathbf{C} = (C, 0, 1, +, \times)$. The automorphisms of $\mathbf{C}$ are the identity mapping and conjugation, which maps any complex number $a + ib$ to its conjugate $a - ib$. So, any complex number and its complex conjugate are structurally indiscernible. The Burgess-Keränen objection may then be formulated as a *reductio ad absurdum*. In $\mathbf{C}$, i and $-i$ are structurally indiscernible. So, by (I), we infer that $i = -i$. This is absurd. More generally, (I) implies that all elements of the same orbit are identical, which is absurd for non-rigid structures. This is by no means an isolated phenomenon, as non-rigid structures abound in mathematics.[5]

An immediate response to this might be to insist that mathematical structuralism is not, despite initial appearances, committed to the indiscernibility principle (I). But, even so, there still remains a more general philosophical problem, concerning the analysis of the notion of identity for the positions of an abstract structure. Must the identity relation on positions be *defined* in terms of the other distinguished relations? Or might the identity relation for positions be taken as *primitive*? For my part, I see no compelling reason why the identity relation, in general, should not be thought of as primitive. The reasons sometimes given for *not* taking identity as primitive seem to me to be anti-realist, reductionist or verificationist in spirit. The contrary view, that identity is primitive and indefinable, was advocated by Gottlob Frege, in his 1891 review of Edmund Husserl's *Philosophie der Arithmetik*: 'since any definition is an identity,

[3] An automorphism of a structure is an isomorphism from the structure to itself. The group of automorphisms of $\mathbf{M}$ is denoted $\text{Aut}(\mathbf{M})$. The identity mapping on the domain is always an automorphism. If the identity mapping is the *only* automorphism of $\mathbf{M}$, then $\mathbf{M}$ is called *rigid*. The ω-sequence $(N, <)$ of natural numbers is rigid, as in fact is any ordinal. In contrast, the integers with their natural order, $(Z, <)$, is non-rigid, since any integer shift is an automorphism.

[4] The existence of structures with non-trivial automorphisms generates a separate puzzle concerning the *referential indeterminacy* in 'picking out' the structurally indiscernible elements (e.g., the roots i and $-i$ in $\mathbf{C}$). This is discussed in Brandom 1996, and further in Field 1998 (in the 2001 reprint, 271–72).

[5] Indeed there are highly symmetric structures where the orbit of any (and thus every) element is the whole domain. For example, any Euclidean space. We say that the automorphism group $\text{Aut}(\mathbf{M})$ 'acts transitively' on the structure $\mathbf{M}$.

identity itself cannot be defined' (see Geach and Black 1980: 80). In an illuminating article, Elias Savellos has similarly argued that 'identity must be viewed as an indefinable, primitive notion' because 'any attempt to define identity is bound to be circular, since the intelligible understanding of the notion of identity must make recourse to the intelligible understanding of identity itself' (1990: 476).

In the present context, one might reply that while such an indefinabilist view might be admissible for objects ordinarily ('non-structurally') conceived, such a view cannot be admitted on a *structuralist* conception of 'positions', whose identity must be constituted by 'intra-structural relations'. Something like this is indeed what Keränen (2001: 327–28) argues. But aside from my remarks in the previous paragraph, I shall avoid this issue entirely below, and instead turn to some of the logical issues surrounding notions of indiscernibility and the definability of identity.

2. *Notions of indiscernibility*

For the discussion below, we need to consider several notions of indiscernibility.[6]

Suppose that L is a first-order language, without function symbols and with finitely many primitive predicate symbols; suppose that $\mathbf{M}$ is an L-structure and that a, $b \in \mathrm{dom}(\mathbf{M})$. Say that a and b are *monadically indiscernible* just in case, for any L-formula $\varphi(x)$, with just x free, $\mathbf{M} \vDash \varphi(a)$ iff $\mathbf{M} \vDash \varphi(b)$. Say that a and b are *polyadically indiscernible* just in case, for any L-formula $\varphi(x, \underline{y})$, $\mathbf{M} \vDash \forall \underline{y}(\varphi(a, \underline{y}) \leftrightarrow \varphi(b, \underline{y}))$, where $\underline{y}$ is a sequence of variables. In 1976, Quine introduced the following notion: a and b are *weakly discernible* in $\mathbf{M}$ just in case there is an irreflexive relation R definable in $\mathbf{M}$ such that Rab.[7] If elements are not weakly discernible, we shall say that they are *strongly indiscernible*.

Finally, first-order indiscernibility.[8] For each n-place symbol $\mathbf{P}$, let s be a sequence $(x, z_1, \ldots, z_{n-1})$ of n distinct variables, and let $\mathbf{P}s$ be the resulting atomic formula. Let y be a new variable, and let $\mathbf{P}s(x/y)$ be the result of substituting y for the occurrence of x in $\mathbf{P}s$. Next, let s' be the sequence $(z_1, x, \ldots, z_{n-1})$, let s'' be $(z_1, z_2, x, \ldots, z_{n-1})$ and so on. Let $x \approx_{\mathbf{P}} y$ be the

[6] Several notions of indiscernibility are also studied in model theory. In contrast with the discussion here, one does not define a binary relation of indiscernibility. Rather, one defines the notions of a *set of indiscernibles* and a *set* of *order-indiscernibles* (see Hodges 1997: 152–53).

[7] Quine uses the phrase 'weakly discriminable'. See Quine 1981: 132. Quine adds that the formulation given here was suggested by Ivan Fox, whereas the original version was slightly different.

[8] See Quine 1960: 230 and Quine 1970: 63–64. The idea goes back to Hilbert and Bernays 1934.

formula $\forall z_1 \ldots \forall z_{n-1}[(\mathbf{P}s \leftrightarrow \mathbf{P}s(x/y)) \wedge (\mathbf{P}s' \leftrightarrow \mathbf{P}s'(x/y)) \wedge \ldots]$. The formula $x \approx_{\mathrm{P}} y$ is the *first-order indiscernibility formula* for the predicate symbol **P**. Next, let $x \approx y$ be the conjunction of the indiscernibility formulae $x \approx_{\mathrm{P}} y$, for each primitive relation symbol **P**. The formula $x \approx y$ is the *first-order indiscernibility formula* for L.[9] For illustration, suppose L has a unary predicate symbol F and a binary predicate symbol G. Then the indiscernibility formula $x \approx y$ is $(Fx \leftrightarrow Fy) \wedge \forall z((Gxz \leftrightarrow Gyz) \wedge (Gzx \leftrightarrow Gzy))$.

The indiscernibility formula $x \approx y$ has those formal properties of identity expressible in first-order logic: i.e reflexivity and substitutivity. One can also show that if the identity relation is first-order definable in a structure at all, then it is defined by the indiscernibility formula.

Write '$\approx_{\mathbf{M}}$' for the *relation* that $x \approx y$ defines on **M**. Clearly, $\approx_{\mathbf{M}}$ is an equivalence relation.[10] In general, the relation $\approx_{\mathbf{M}}$ may, or may not, be the identity relation on the domain of **M**. Let us say that a structure **M** is *Quinian*[11] just in case $\approx_{\mathbf{M}}$ is the identity relation: that is, just in case, for all a, $b \in \mathrm{dom}(\mathbf{M})$, if $a \approx_{\mathbf{M}} b$, then $a = b$. Quinian structures are precisely those in which the identity relation is first-order definable.[12]

Monadic indiscernibility is the weakest of these notions; polyadic, strong and first-order indiscernibility yield the strongest notions expressible in first-order logic, and in fact are equivalent. Note that all of these notions are model/language relative. So, if elements of **M** are also elements of **M′**, they might be indiscernible in **M**, but discernible in **M′**.

3. *Ladyman's proposal: weak discernibility*

James Ladyman (2005) has presented a response to the Burgess-Keränen objection. Following Simon Saunders (2003), who applies a similar proposal to the case of physical entities, Ladyman proposes an indiscernibility

[9] The formula $x \approx y$ depends upon the choice of variables $z_1, \ldots, z_{n-1}$. But, up to logical equivalence, it doesn't matter which variables are chosen, so long as they are distinct from each other and from x and y. If L contains the identity predicate, $x \approx y$ becomes $\ldots \wedge \forall z((x = z \leftrightarrow y = z) \wedge (z = x \leftrightarrow z = y)) \wedge \ldots$, which reduces (in first-order logic with rules for identity) to $x = y$.

[10] Since $\approx_{\mathbf{M}}$ is an equivalence relation, we may consider the quotient structure $\mathbf{M}/{\approx_{\mathbf{M}}}$, which, by construction, is 'Quinian' on the definition given below. In general, **M** and $\mathbf{M}/{\approx_{\mathbf{M}}}$ are elementarily equivalent.

[11] Some authors write 'Quinean'. But Quine himself adjectivizes his own name only once in his works, so far as I know; and he uses the term 'Quinian'. See Quine 1960: 171: '... any more than there need be some peculiarly Quinian textural quality common to the protoplasm of my head and feet'.

[12] Quine noted, in effect, that there exist non-Quinian structures. See Quine 1970: 63, 'it may happen that the objects intended as values of variables of quantification are not completely distinguishable from one another ...'.

principle based on weak discernibility, and suggests that this resolves the objection given by Burgess and Keränen:

> The structuralist can thereby explain the manifest non-identity within mathematics of such entities without violating the Identity of Indiscernibles, by adopting the version of the weakest form of the Identity of Indiscernibles which demands only weak, and not strong or relative, discernibility of numerically distinct individuals. (Ladyman 2005: 220)

The indiscernibility principle that Ladyman has in mind is that 'numerically distinct individuals' should be weakly discernible. Contraposing this yields:

(II) For any structure **M**, if $a, b \in \mathrm{dom}(\mathbf{M})$ are strongly indiscernible in **M**, then $a = b$.

Now strong and first-order indiscernibility are equivalent, so (II) is equivalent to

(III) For any structure **M**, for all $a, b \in \mathrm{dom}(\mathbf{M})$, if $a \approx_{\mathbf{M}} b$, then $a = b$.

Finally, using our definitions, (III) is equivalent to

(IV) All structures are Quinian.

This is a rather strong claim, and we shall see in a moment that there are counter-examples.

Ladyman considers the case of the complex field, and notes (in effect) that **C** is Quinian. Indeed, i and $-i$ are weakly discernible in the complex field **C**, since $-i$ is the additive inverse of i. And the relation expressed by '$x \neq 0$ & x is the additive inverse of y' is irreflexive.[13] This is correct. Indeed, weak discernibility of distinct elements holds for many structures used in mathematics and mathematical physics (e.g. groups, fields, topological spaces).

4. *A preliminary objection*

There is an immediate objection. The question of weak discernibility for structures like **C** is entirely trivial. For 'x is the additive inverse of y' is expressed by the formula $y + x = 0$, and one needn't be Sherlock Holmes to observe that this contains the *identity predicate*. Moreover, in order to express that x is *the* additive inverse of y, one needs *uniqueness*, defined using the identity predicate. Thus, the fact that i and $-i$ are weakly discernible in the field **C** is trivial: they are weakly discerned by $x \neq y$.

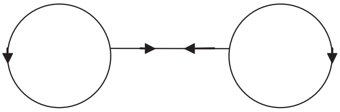
[13] Note that 'x is the additive inverse of y' is not irreflexive, since 0 is the additive inverse of itself. This is a minor quibble, easily fixed by considering the relation defined by '$x \neq 0$ and x is the additive inverse of y'.

More generally, any field is an algebraic structure, for which the identity relation is assumed as a primitive. The atomic formulae of the language of any field are *equations*, of the form $t = u$, where t and u are terms. In general, the notion of an algebraic structure is specified using a language with a *primitive identity predicate*, along with function symbols ('+', '×', etc.).[14] Such structures are specified in terms of functions, thereby presupposing the notion of identity. In short, the notion of identity is *presupposed* in the specification – whether by direct construction or by axiomatization – of algebraic structures: algebraic structures are Quinian *by construction*.

If this objection is right, then there is no 'identity problem' for structuralism in connection with algebraic structures. For the primitive notion of identity is *presupposed* in specifying what a group, field, etc., *is*. More generally, there is no 'identity problem' for structuralism in connection with *Quinian* structures. Since for these, although identity might not be taken as a primitive distinguished relation, the identity relation is nonetheless *definable*.

5. *A counter-example to Ladyman's Indiscernibility Principle*

On Ladyman's view, the structuralist should be committed to (II), which is equivalent to the claim (IV) that all structures are Quinian. But there are counter-examples to this claim. Consider the two-element 'dumb-bell' structure, pictured thus:



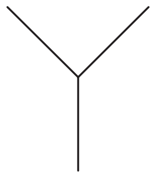
This structure has two structurally indiscernible 'positions'. But these positions are also not first-order discernible, for the only irreflexive relation definable in the dumb-bell is the empty relation. According to (II),

[14] Keränen appears to dispute this, and writes, 'the language of the theory of groups is just {+, –, 0}' (2001: 319). But this is not right. The language of the theory of groups contains the identity predicate as a primitive. A group **G**, by definition, is an algebraic structure (D, ·) such that · is an associative binary operation, and there is an element e such that, for any element a, $a \cdot e = a = e \cdot a$, and for each a, there is an element b such that $a \cdot b = e = b \cdot a$. *How* could these defining conditions be *expressed* without using the notion of identity? If one attempts to define a group **G** as a *relational* structure (D, R) where R is a ternary relation, such that certain conditions on D and R hold, then one can show that no such conditions can be given if the notion of identity is disallowed.

Ladyman's mathematical structuralist must conclude that the dumb-bell has *one* position, which is absurd.[15] The structure is non-Quinian.

6. *Definability of identity in relational structures*

We have seen that there exist non-Quinian structures. How do we find non-Quinian structures? One way to show that that identity is not definable is to find two distinct positions which are first-order indiscernible. The dumb-bell structure yields such an example, as does the partial ordering below:



The two maximal elements are indiscernible. So, identity is not definable in this structure.

Visualizing a structure is not always a good guide to indiscernibility. Our perceptual and cognitive mechanisms seem to be good at detecting *symmetries* of visually presented diagrams: i.e. automorphisms. But the fact that elements are related by an automorphism does not mean that they are indiscernible. An example is the symmetric structure, —←——→—. Swapping the end-points is an automorphism, but the end-points are discernible.[16]

There is a characterization problem for Quinian structures. Under what conditions is a structure Quinian?

Theorem: Each of the following conditions is sufficient for **M** to be Quinian:

(i) A surjective total function is definable in **M**.
(ii) A total order on the domain is definable in **M**.
(iii) **M** is rigid.

Proof: For (i), suppose a surjective total function f is definable, by $\varphi(x, y)$ say. So, for any elements a, b, we have $\mathbf{M} \vDash \varphi(a, b)$ iff $b = f(a)$. Then the formula $\exists z(\varphi(z, x) \wedge \varphi(z, y))$ defines the identity relation. For (ii), suppose

[15] Button (2006: 218) presents the same counter-example. Manzano (1996: 55) and Keränen (2001: 321) also give examples of a non-Quinian structures.

[16] If we call the elements 0 and 1, then the relation R is $\{(0, 1), (1, 0)\}$. So, 0 is discernible from 1, since $(0, 1) \in R$ but $(1, 1) \notin R$. That is, 0 has the property of being related to 1, but 1 does not have this property.

a total order is definable, by $\varphi(x, y)$ say. Then the formula $\neg\varphi(x, y) \wedge \neg\varphi(y, x)$ defines identity. For (iii), suppose the identity relation is not definable. So, there are distinct a, b such that $a \approx_{\mathbf{M}} b$. Let π_{ab} be the bijection of the domain which swaps a and b and leaves all other elements alone. It can be shown in general that if $a \approx_{\mathbf{M}} b$, then π_{ab} is an automorphism of **M**. (The converse of this result does not hold in general: the symmetric structure mentioned above is a counter-example.) So, π_{ab} is an automorphism. But since a and b are distinct, π_{ab} is a non-trivial automorphism, and thus **M** is not rigid.

Condition (i) is also necessary, since identity is a surjective total function. However, conditions (ii) and (iii) are not necessary.[17]

7. *Must identity be definable?*

It is unclear to me why mathematical structuralism should require that *ante rem* structures be Quinian, or that a primitive notion of identity should be somehow inadmissible. Why should the identity relation on the domain of a structure be definable from the *other* distinguished relations? Identity *is* a binary relation, although a rather special one.

In any case – and with a certain caveat – abandoning the primitive identity predicate would have dramatic consequences for even the most elementary parts of mathematics.[18] Without the identity predicate, one cannot specify the usual kinds of structures that are the bread-and-butter of ordinary mathematics. Certain mathematical notions presuppose identity: for example, the notions of *uniqueness*, *function* and *finite cardinality*.

Consider the usual way to express that a relation is a function. For a binary relation symbol F we *presuppose identity*, and set down the axiom $\forall x \forall y \forall z(Fxy \wedge Fxz \rightarrow y = z)$. If a structure satisfies this, then the relation denoted by F must be a function. However, if we drop the identity predicate, and replace $x = y$ by the indiscernibility formula $x \approx y$ (namely $\forall z(Fxz \leftrightarrow Fyz) \wedge \forall z(Fzx \leftrightarrow Fzy)$), we get the axiom of 'pseudo-functionality': $\forall x \forall y \forall z(Fxy \wedge Fxz \rightarrow y \approx z)$. However, there are structures which satisfy this, but where the relation denoted by F is *not* a function. For

[17] For example, the complex field **C** is Quinian but non-rigid. Moreover, one cannot define a total order < in **C**. The proof is this: any relation definable in a structure is invariant under any automorphism. If a total order < were definable in **C**, then we could infer that $-i < i$ iff $(-i)^* < i^*$, iff $i < -i$. And thus either $\neg(-i < i)$ & $\neg(i < -i)$, or both $-i < i$ & $i < -i$. Since < is total, the first disjunct implies that $i = -i$, which is absurd. Since < is transitive, the second disjunct implies that $-i < -i$, which is also absurd.

[18] The caveat is that mathematical reality 'as a whole' might permit the definability of identity. For example, identity *is* definable in the cumulative hierarchy **V**, using extensionality: $\forall x \forall y(\forall z(z \in x \leftrightarrow z \in y) \rightarrow x = y)$.

example, the dumb-bell structure above satisfies pseudo-functionality, but the primitive relation is not a function.

More generally, if one drops the primitive identity predicate, one cannot specify that a structure is an algebraic structure (i.e group, a field, etc.). Similarly, one cannot define the standard model of arithmetic, since one cannot constrain the successor relation to be a function. Similarly, one cannot specify that a structure has at most n elements, for finite n. For example, to express that a structure has *at most one element*, we use the axiom $\forall x \forall y(x = y)$. However, replacing $x = y$ by $x \approx y$, we get $\forall x \forall y(x \approx y)$. But the dumb-bell structure satisfies this formula, even though it has *two* elements.

Simon Saunders has claimed that the analysis of identity in terms of first-order indiscernibility is 'the *only* analysis of identity that is really workable from a logical point of view' (Saunders 2003: 292). To see why this is incorrect, consider again the dumb-bell structure, whose theory can be axiomatized by the formulae $\forall x \forall y Fxy$ and $\exists x \exists y(x \neq y \wedge \forall z(z = x \vee z = y))$. Any model of these is isomorphic to the dumb-bell.[19] The cardinality axiom contains the identity predicate. Replacing = by $\approx$ yields the formula $\exists x \exists y(\neg(x \approx y) \wedge \forall z(z \approx x \vee z \approx y))$. But this is *false* in the dumb-bell, because $\exists x \exists y \neg(x \approx y)$ is false.

Indeed, there is no theory T in the first-order language without identity, with sole binary predicate symbol F, such that the models of T are precisely those isomorphic to the dumb-bell.[20] There *is*, as we have seen, such a theory in the language with identity. That is, first-order logic with identity is more expressive than first-order logic without identity. There *is* also a theory in the *second-order* language without identity, but with binary predicate symbol F, and whose *full* models are those required. But this requires the second-order Principle of Identity of Indiscernibles that Saunders rejects.

So, for non-Quinian structures, the first-order 'analysis of identity' proposed by Saunders is mathematically unworkable. In general, for a non-Quinian structure, some of its identity facts must be 'primitive' identity facts, in the technical sense that the identity relation cannot be defined in other terms.

[19] Let **M** be a model of these axioms. By the cardinality axiom, **M** must contain exactly two elements, say a and b. Because $\forall x \forall y Fxy$ is true, a and b must be related to themselves and each other. So, the relation that F denotes must be $\{(a, a), (a, b), (b, a), (b, b)\}$. This is an instance of the dumb-bell structure.

[20] Suppose T were such a theory. It would contain the formula $\forall x \forall y Fxy$. But this axiom is complete in predicate logic without identity (any two models are elementarily equivalent). So $\forall x \forall y Fxy$ would *axiomatize* **T**. The structure **M** with domain $\{a\}$ and relation $\{(a, a)\}$ is a model of T, but it is *not* an instance of the dumb-bell.

8. *The weakest form of the Identity of Indiscernibles?*

Ladyman says of (II), and thus of its equivalent (III), that it is the 'weakest form of the Identity of Indiscernibles'. But this is not quite the case. (III) is indeed the weakest *first-order* identity principle. But *weaker* still is the second-order Principle of Identity of Indiscernibles:

(PII) $\forall X(Xx \rightarrow Xy) \rightarrow x = y$.

There is a well-established usage according to which the Principle of Identity of Indiscernibles really is the above *second-order* principle.[21] And, as we have seen, any *first-order* indiscernibility principle will have counter-examples (i.e non-Quinian structures).

If one is considering abandoning (PII), it should also be stressed that (PII) is a *theorem* in any deductive system (for second-order logic) which includes the comprehension axiom:

(Comp) $\forall y_1 \dots \forall y_n \exists X \forall x[Xx \leftrightarrow \varphi(x, y_1, \dots, y_n)]$.

Take the instance

(Haec) $\forall y \exists X \forall x(Xx \leftrightarrow x = y)$.

This asserts, for any object a, the existence of the *property of being a*, sometimes called the *haecceity* of a. The extension of this property is the unit set $\{a\}$. So, (Haec) can be read as saying 'for any object a, the haecceity of a exists'.

The proof of (PII) using (Haec) is as follows. Suppose (PII) is false, and there are a, b such that $a \neq b$ but $\forall X(Xa \rightarrow Xb)$. By (Haec), we infer that $\exists X \forall x(Xx \leftrightarrow x = a)$. Call this property P. So, $Px \leftrightarrow x = a$. So, $Pa \leftrightarrow a = a$, and thus Pa. But $Pa \rightarrow Pb$. Thus, Pb; and thus $b = a$. Thus, by the usual substitution rule for identity, $a \neq a$. Contradiction.

Any full (standard) second-order structure satisfies (PII), and is thus Quinian in the second-order sense: the Leibniz-Russell indiscernibility formula $\forall X(Xx \rightarrow Xy)$ defines the identity relation. And we have seen that any Henkin (or general) structure which satisfies (Haec) also satisfies (PII).

However, one *can* find counter-models for (PII) if one considers Henkin structures with 'missing haecceities'. A Henkin structure for (monadic) second-order logic is one where the second-order variables may range over a suitable proper subset S of the power set of the first-order domain. For example, we can treat the dumb-bell structure **M** mentioned above as a Henkin structure, where, for definiteness, we take the first-order domain to be $\{0, 1\}$. Let S be the collection of sets definable in **M**. Then $S = \{\emptyset, \{0, 1\}\}$. So, the unit sets $\{0\}$ and $\{1\}$ are *not definable*. These unit sets are the

[21] See, e.g., van Dalen 1994: 151–52; Manzano 1996: 2: 53–55; Shapiro 1991: 63; Boolos, Burgess and Jeffrey 2002: 280.

'missing haecceities'. Consider the Henkin structure ($\mathbf{M}, S$). Although 0 and 1 are distinct (*ex hypothesi*), the formula $\forall X(Xx \rightarrow Xy)$ identifies them. This non-full Henkin structure is thus non-Quinian in a second-order sense.

9. *A final thought: ontological rigidity?*

Let me conclude with a metaphysical fantasy. Let us assume for the moment that there is such a thing as 'the structure' of reality, and let us indulge in the fantasy that this structure is a first-order structure (a domain with some finite collection of privileged relations). Consider the claim that *the structure of reality is rigid*. Call this doctrine *ontological rigidity*. That is, although 'fragments' of reality exhibit symmetries (permutation symmetries, gauge symmetries, various symmetries that lead to conservation laws, etc.), 'reality as a whole' has no non-trivial symmetries. I have no idea whether ontological rigidity is true, or even whether it is coherent. Perhaps it provides a way of reading Bishop Butler's anti-reductionist remark, that 'every thing is what it is, and not another thing'.[22] In any case, if ontological rigidity holds, we may apply condition (iii) from the theorem above and infer that the world is Quinian.[23]

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[22] Butler's widely-cited remark is from Butler 1726, *Fifteen Sermons*, Preface, paragraph 39: 'If the observation be true, it follows, that self-love and benevolence, virtue and interest, are not to be opposed, but only to be distinguished from each other; in the same way as virtue and any other particular affection, love of arts, suppose, are to be distinguished. Every thing is what it is, and not another thing.'

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