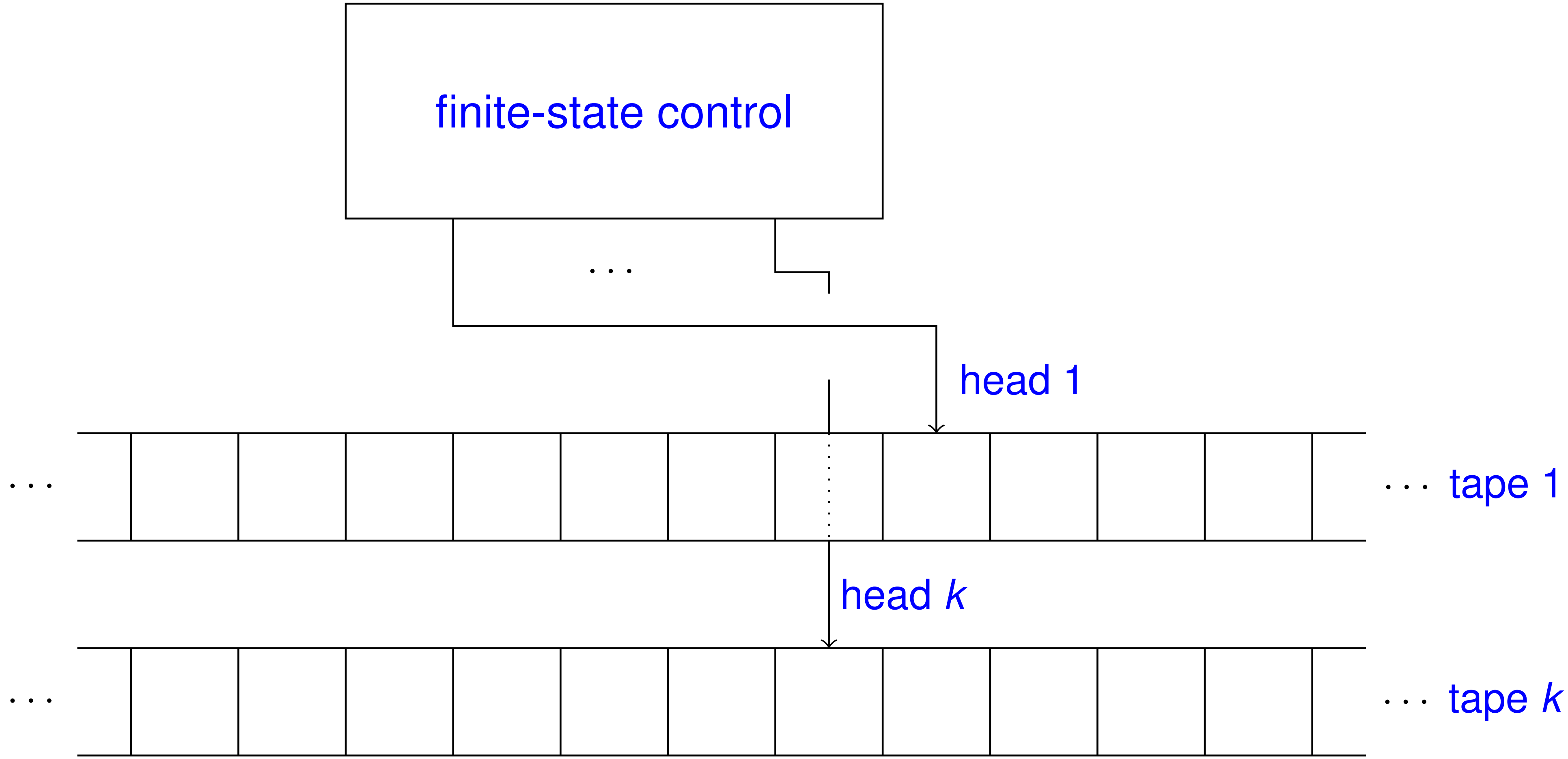
Chapter 4:

Turing Machines and Recursively Enumerable and Context-Sensitive Languages

4.1. Turing Machines

The Turing machine is a mathematical model of a computing device.
Informally, it looks as follows:



Formally a Turing machine (TM) with $k \geq 1$ tapes is defined through a 7-tuple $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$, where

- Q is a finite set of (internal) states,
- Σ is a finite input alphabet,
- $\Gamma \supsetneq \Sigma$ is a finite tape alphabet,
- $\Box \in \Gamma \setminus \Sigma$ is the blank symbol,
- $\delta : Q \times \Gamma^k \rightsquigarrow Q \times \Gamma^k \times \{L, 0, R\}^k$ is the transition function,
- $q_0 \in Q$ is the initial state, and
- $q_1 \in Q$ is the halting state.

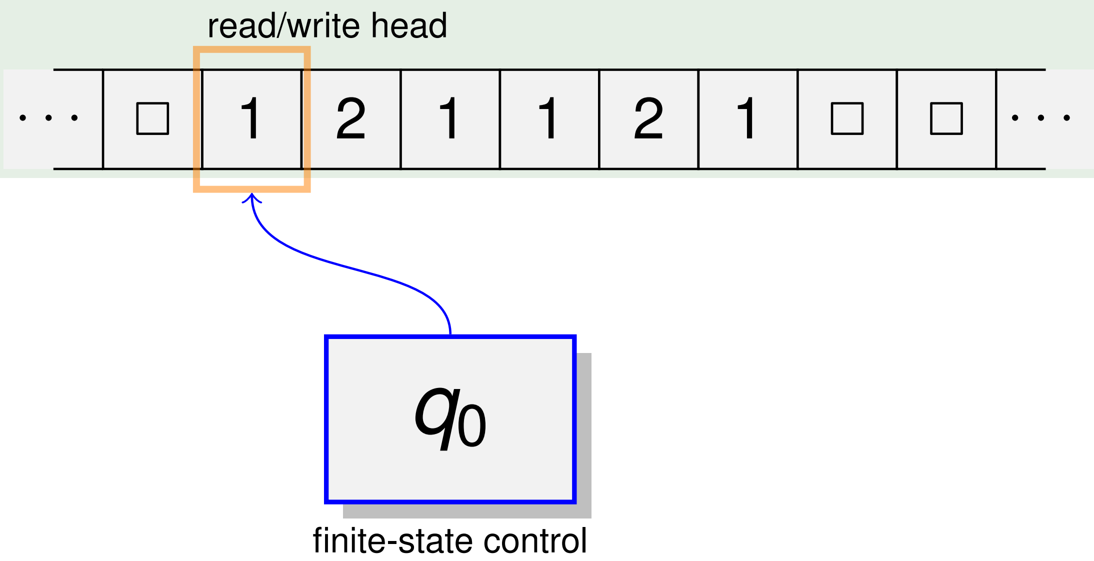
A TM works in discrete steps, where each step consists of 4 parts:

1. read current symbols under the heads,
2. write new symbols using the heads,
3. move the heads,
4. change the state.

Example:

A TM $M = (\{q_0, \ldots, q_3\}, \{1, 2\}, \{1, 2, \Box\}, \Box, \delta, q_0, q_1)$ for computing the function $+1$ on dyadic presentations:

$$
\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
q_0 1 & \to & q_0 1 R & q_2 1 & \to & q_2 1 L & q_3 1 & \to & q_2 2 L \\
q_0 2 & \to & q_0 2 R & q_2 2 & \to & q_2 2 L & q_3 2 & \to & q_3 1 L \\
q_0 \Box & \to & q_3 \Box L & q_2 \Box & \to & q_1 \Box R & q_3 \Box & \to & q_1 10
\end{array}
$$



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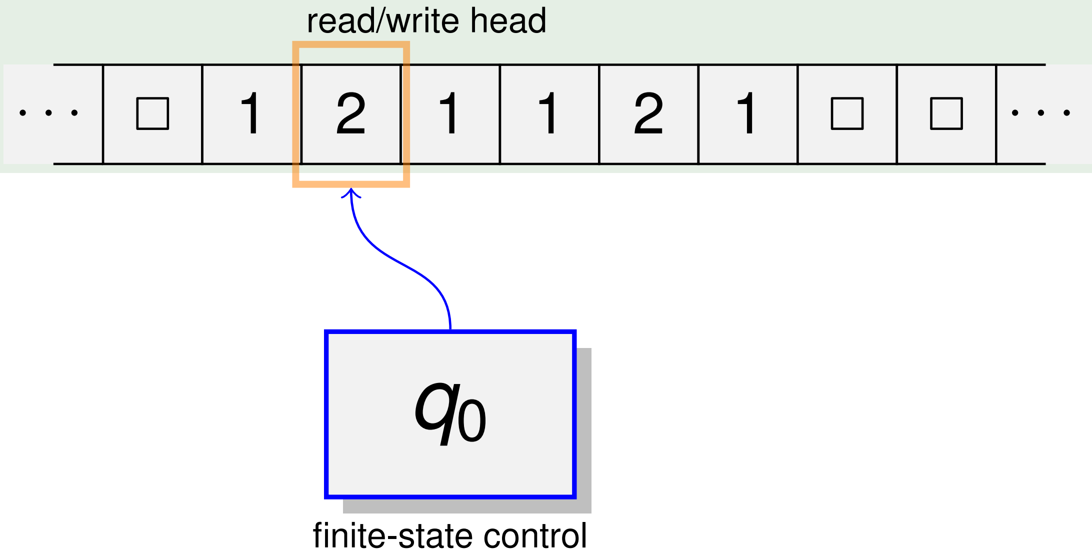

Example:

A TM $M = (\{q_0, \ldots, q_3\}, \{1,2\}, \{1,2,\square\}, \square, \delta, q_0, q_1)$ for computing the function $+1$ on dyadic presentations:

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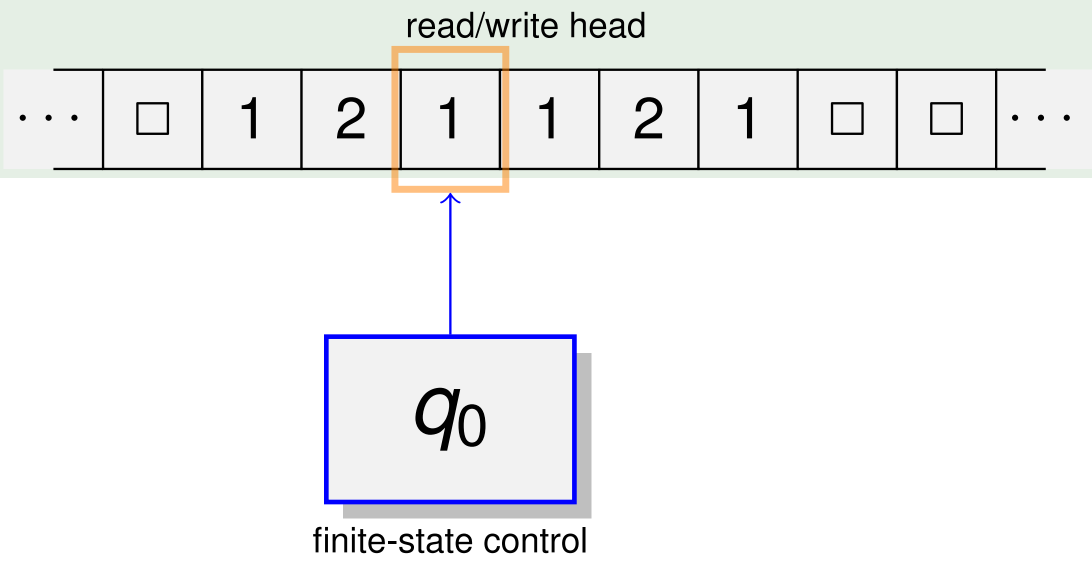

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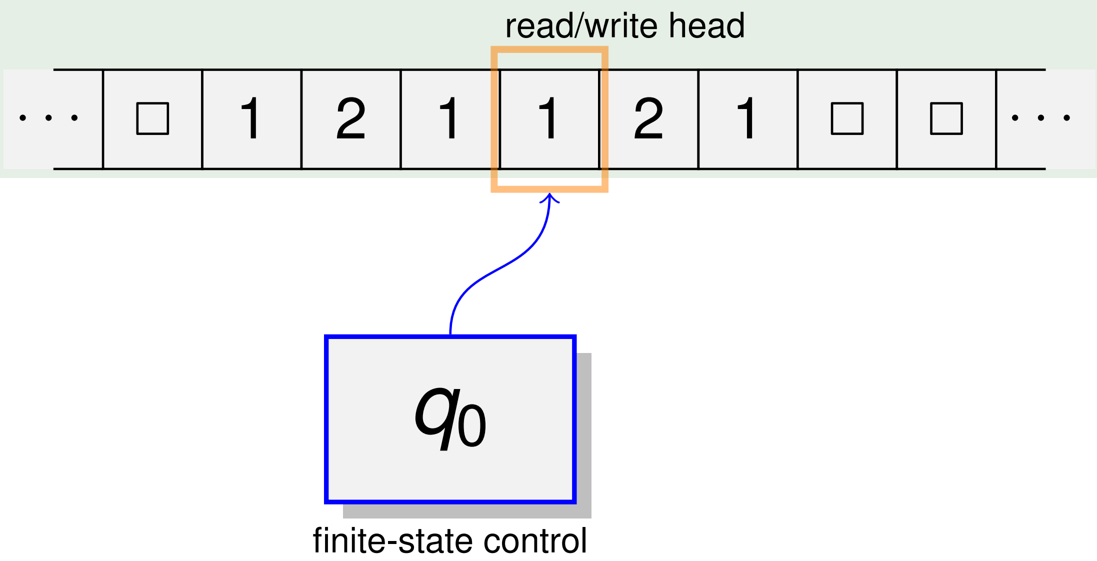
$q_0 2$	$\rightarrow$	$q_0 2 R$	$q_2 2$	$\rightarrow$	$q_2 2 L$	$q_3 2$	$\rightarrow$	$q_3 1 L$
$q_0 \Box$	$\rightarrow$	$q_3 \Box L$	$q_2 \Box$	$\rightarrow$	$q_1 \Box R$	$q_3 \Box$	$\rightarrow$	$q_1 10$



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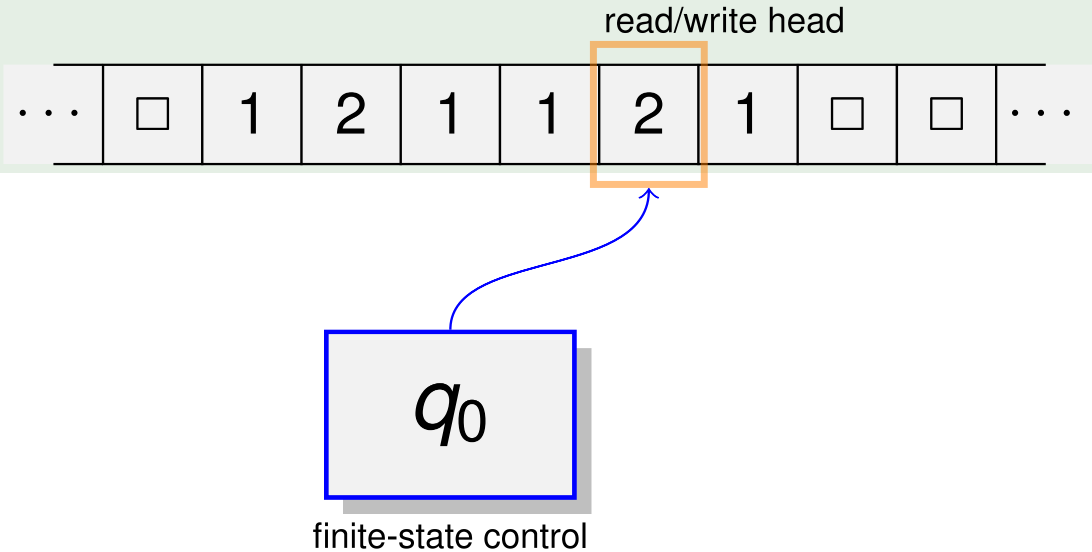

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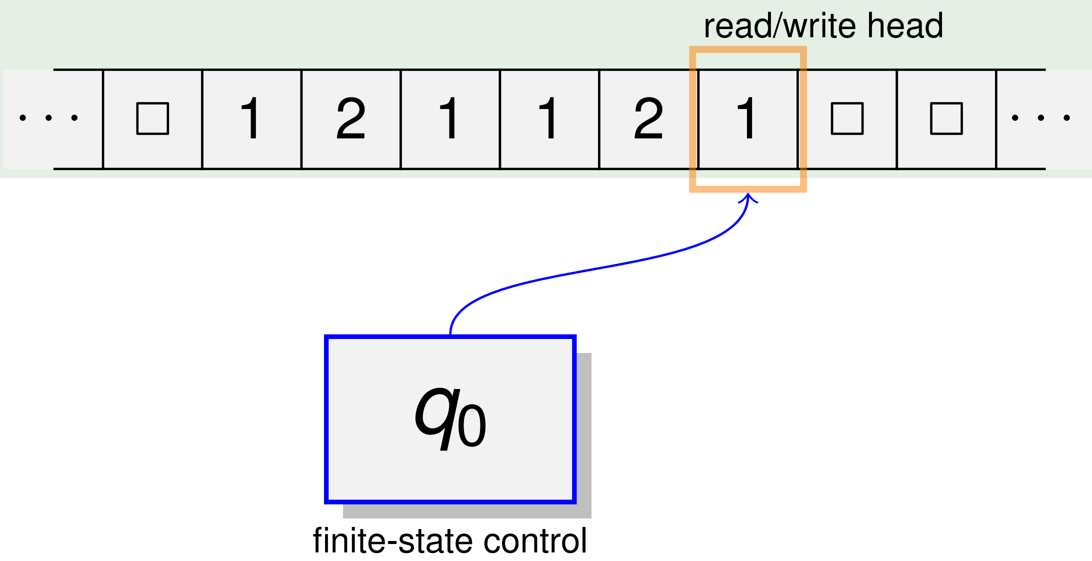

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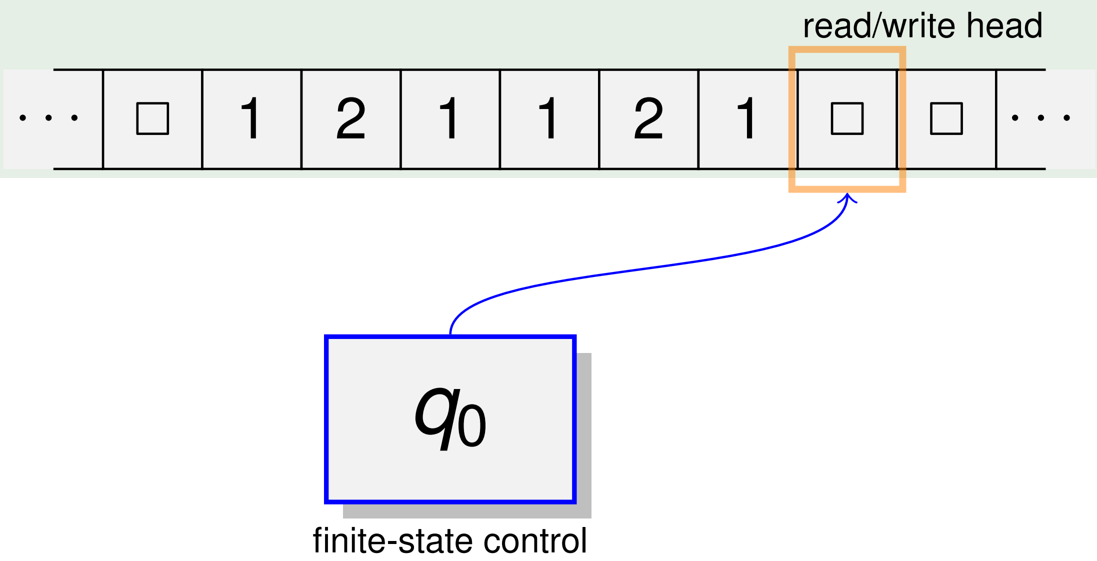

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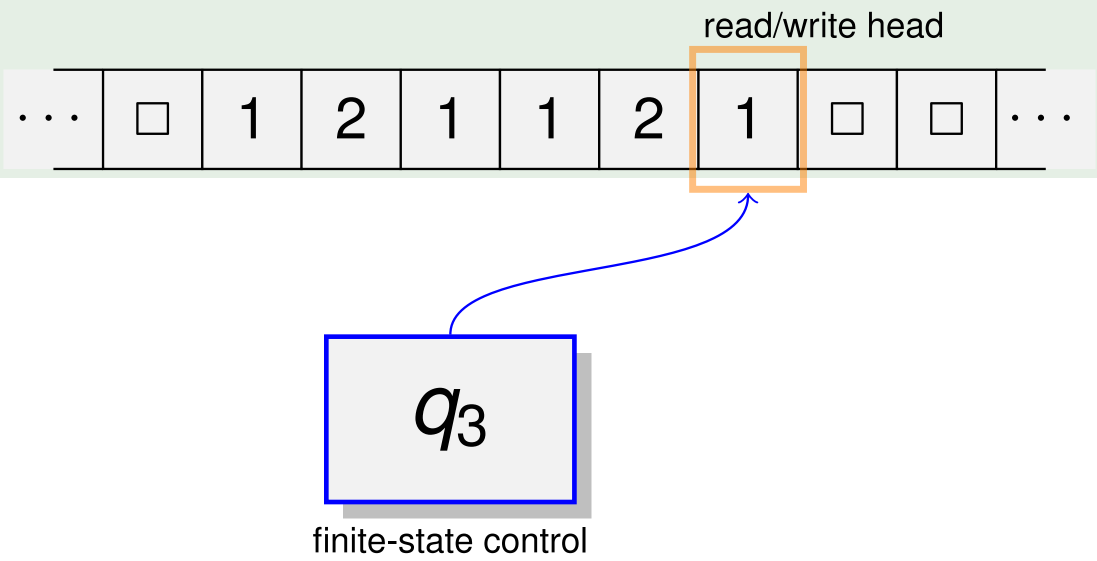

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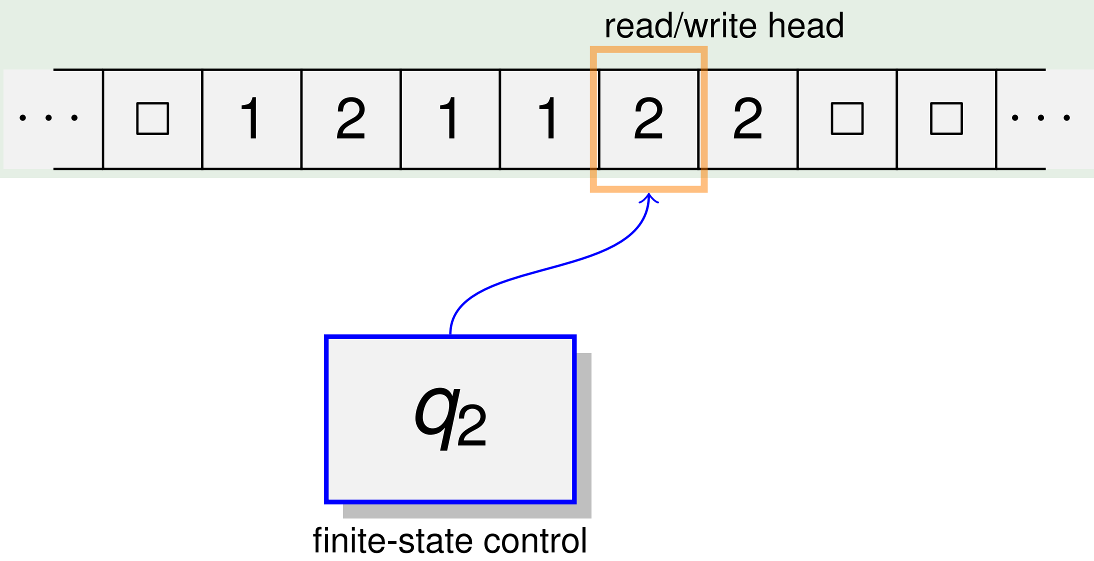

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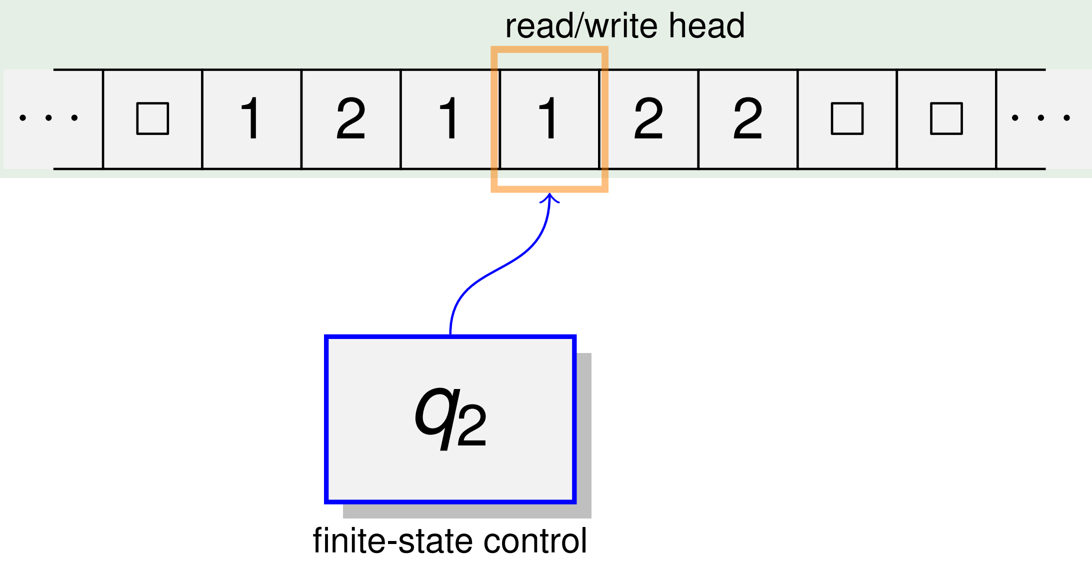

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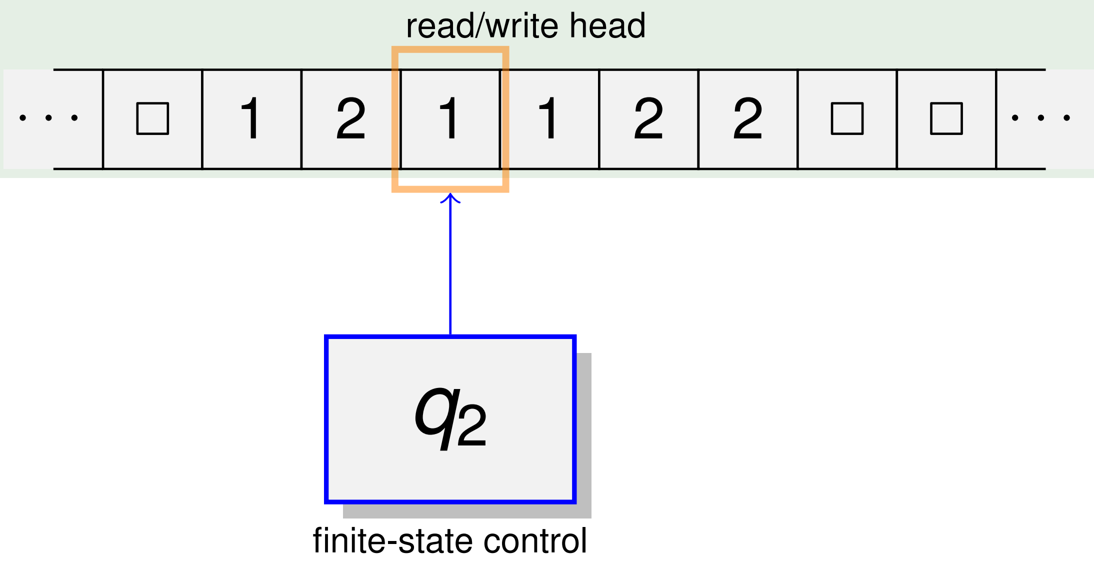

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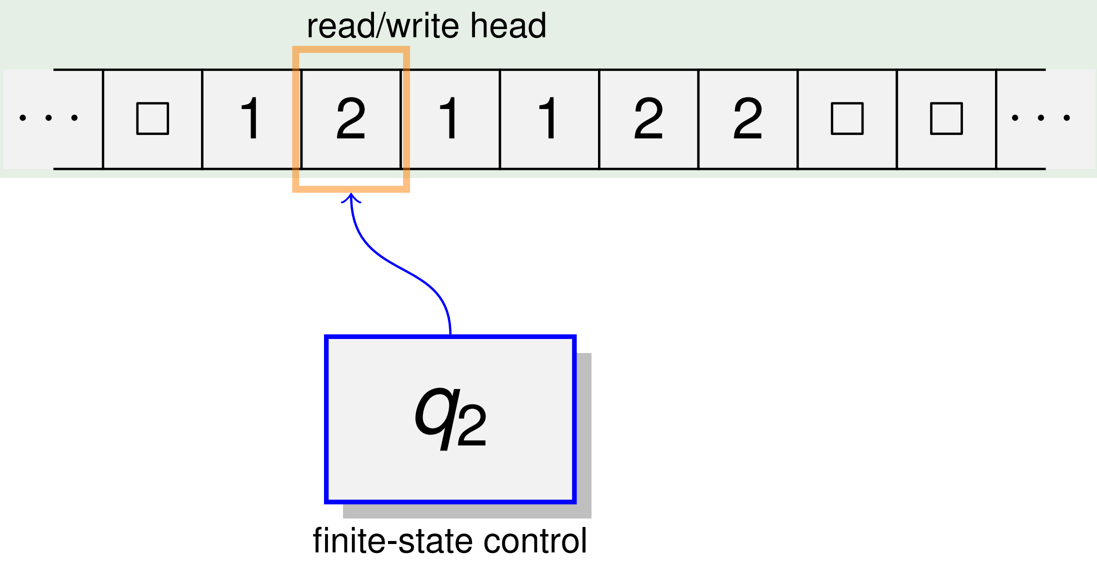

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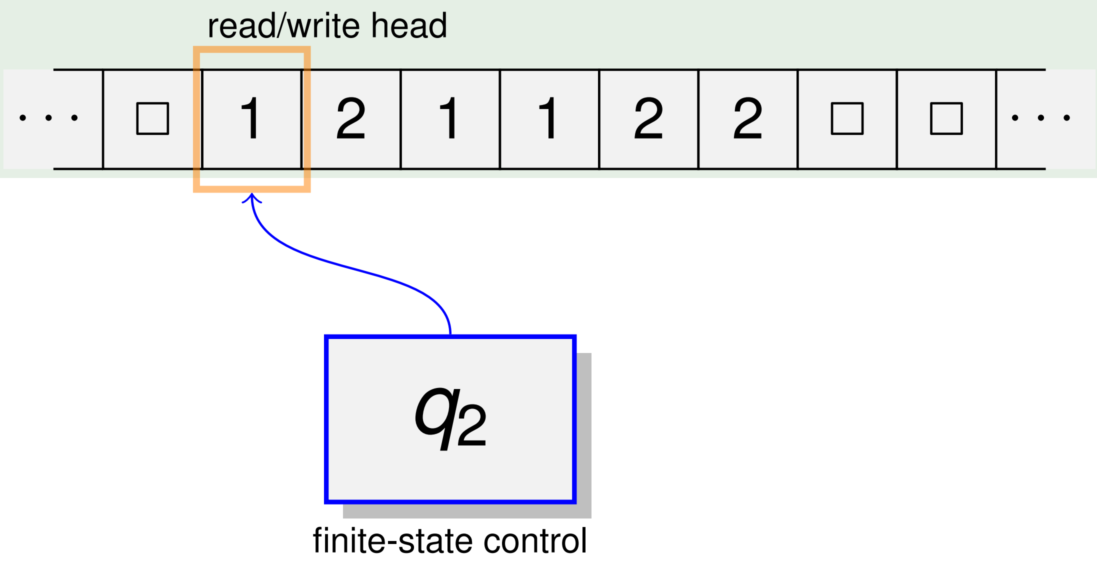

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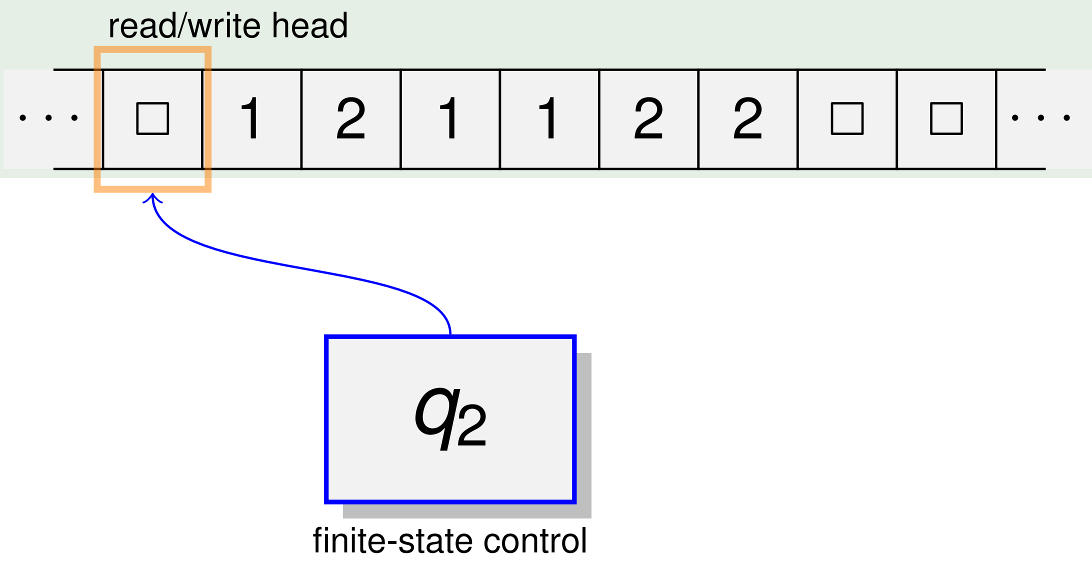

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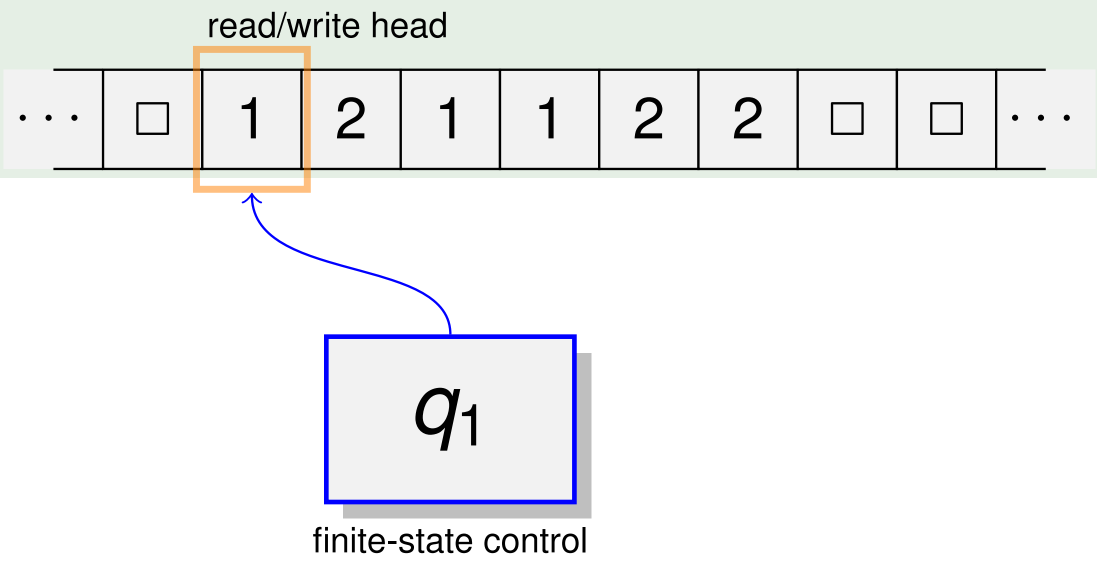
$q_0 \Box$	$\to$	$q_3 \Box L$	$q_2 \Box$	$\to$	$q_1 \Box R$	$q_3 \Box$	$\to$	$q_1 10$



Example:

A TM M for deciding membership in $\{ w \in \{a, b\}^* \mid w = w^R \}$.

On input $w = abaabaaba$, the TM M starts with the following tape contents:

Start:

$\cdots$	$\Box$	$\downarrow$ a	b	a	a	b	a	a	b	a	$\Box$	$\cdots$

Let $\Gamma = \{\Box, a, b\}$ and $Q = \{q_a, q_b, q'_a, q'_b, q_0, q_1, q_2, q_3\}$, where these states are to be used as follows:

q_a (q_b): Remember a (b) and move right.

q'_a (q'_b): Make one step to the left and test for a (b).

q_0 : Start and remember the first letter.

q_1 : Halt.

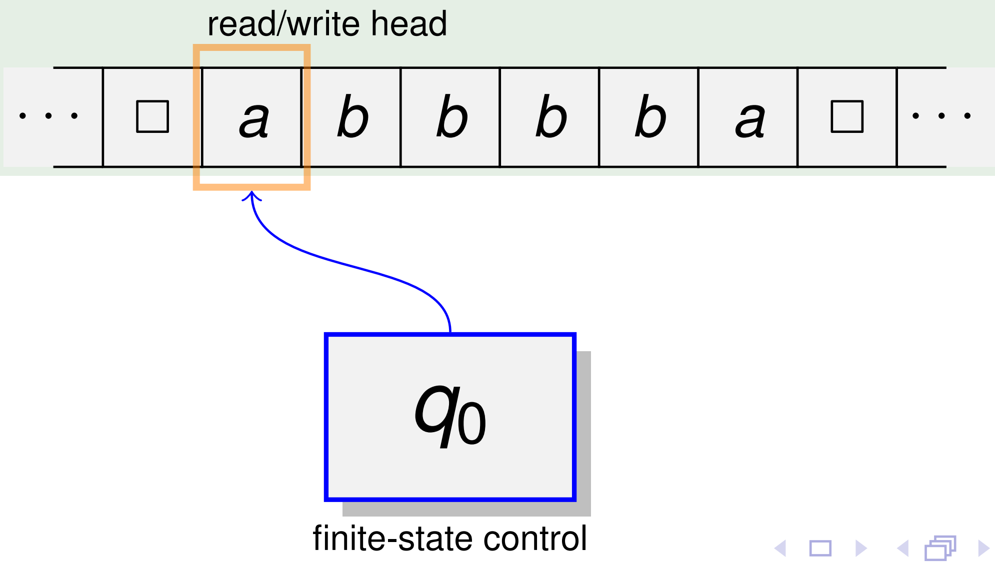
q_2 : Test was positive, return to the left.

q_3 : Test was negative, move left, erase the tape, and halt.

Example (cont.):

The transition function δ is defined as follows:

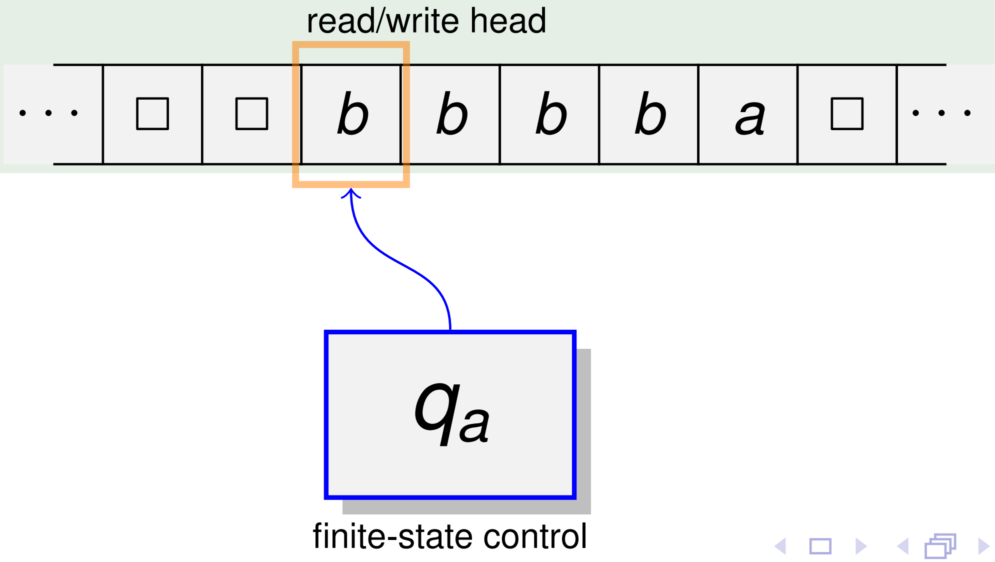
$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
q_0\square & \to & q_1\, a\, 0 & q'_a\square & \to & q_1\, a\, 0 & q_2 b & \to & q_2\, b\, L \\
q_0 a & \to & q_a \square R & q_b a & \to & q_b\, a\, R & q_2\square & \to & q_0 \square R \\
q_0 b & \to & q_b \square R & q_b b & \to & q_b\, b\, R & q_3 a & \to & q_3 \square L \\
q_a a & \to & q_a\, a\, R & q_b\square & \to & q'_b \square L & q_3 b & \to & q_3 \square L \\
q_a b & \to & q_a\, b\, R & q'_b a & \to & q_3 \square L & q_3\square & \to & q_1\, b\, 0 \\
q_a\square & \to & q'_a \square L & q'_b b & \to & q_2 \square L & & & \\
q'_a a & \to & q_2 \square L & q'_b\square & \to & q_1\, a\, 0 & & & \\
q'_a b & \to & q_3 \square L & q_2 a & \to & q_2\, a\, L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

$q_0\Box$	$\to$	$q_1\,a\,0$	$q'_a\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\,\Box\,R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\,\Box\,R$
$q_0 b$	$\to$	$q_b\,\Box\,R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\,\Box\,L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q'_b\,\Box\,L$	$q_3 b$	$\to$	$q_3\,\Box\,L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\,\Box\,L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q'_a\,\Box\,L$	$q'_b b$	$\to$	$q_2\,\Box\,L$			
$q'_a a$	$\to$	$q_2\,\Box\,L$	$q'_b\Box$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\,\Box\,L$	$q_2 a$	$\to$	$q_2\,a\,L$			



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q_0 a \to q_a\,\Box\,R & q_b a \to q_b\,a\,R & q_2\Box \to q_0\,\Box\,R \\
q_0 b \to q_b\,\Box\,R & q_b b \to q_b\,b\,R & q_3 a \to q_3\,\Box\,L \\
q_a a \to q_a\,a\,R & q_b\Box \to q'_b\,\Box\,L & q_3 b \to q_3\,\Box\,L \\
q_a b \to q_a\,b\,R & q'_b a \to q_3\,\Box\,L & q_3\Box \to q_1\,b\,0 \\
q_a\Box \to q'_a\,\Box\,L & q'_b b \to q_2\,\Box\,L & \\
q'_a a \to q_2\,\Box\,L & q'_b\Box \to q_1\,a\,0 & \\
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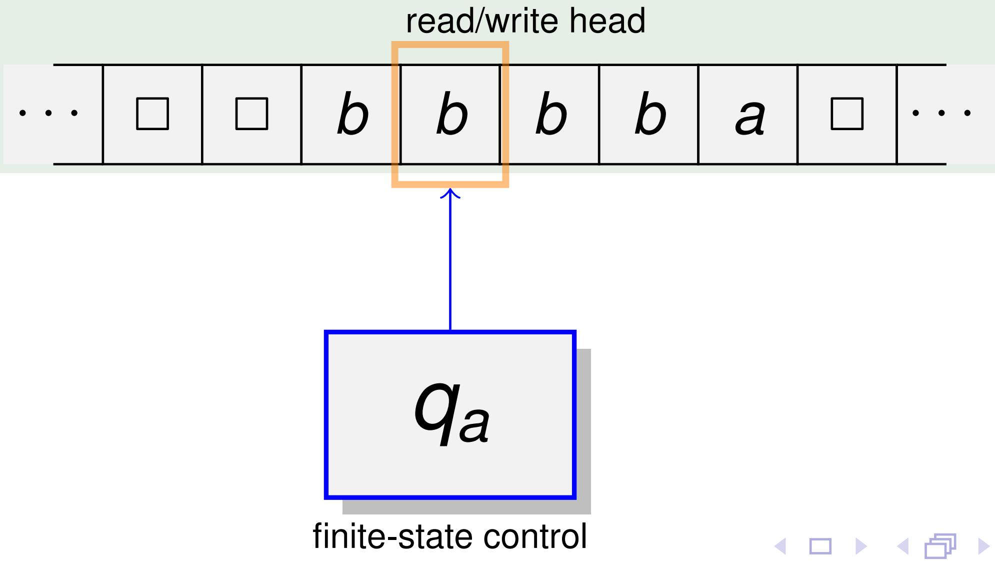

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$q_0 a \to q_a\,\Box\,R$	$q_b a \to q_b\,a\,R$	$q_2\Box \to q_0\,\Box\,R$
$q_0 b \to q_b\,\Box\,R$	$q_b b \to q_b\,b\,R$	$q_3 a \to q_3\,\Box\,L$
$q_a a \to q_a\,a\,R$	$q_b\Box \to q'_b\,\Box\,L$	$q_3 b \to q_3\,\Box\,L$
$q_a b \to q_a\,b\,R$	$q'_b a \to q_3\,\Box\,L$	$q_3\Box \to q_1\,b\,0$
$q_a\Box \to q'_a\,\Box\,L$	$q'_b b \to q_2\,\Box\,L$	
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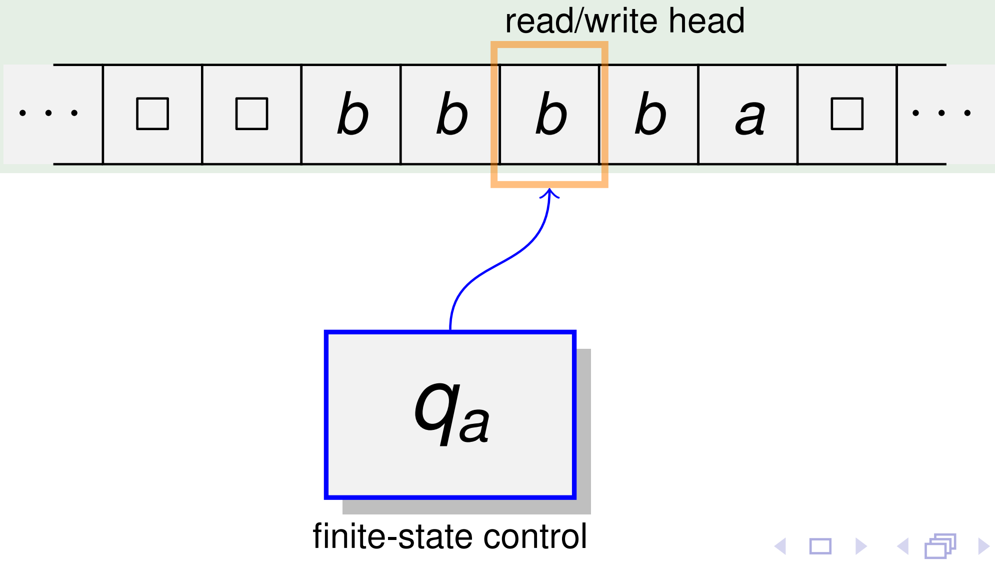

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q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q'_b\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q'_b a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q'_a\Box L & q'_b b & \to & q_2\Box L & & & \\
q'_a a & \to & q_2\Box L & q'_b\Box & \to & q_1\,a\,0 & & & \\
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\end{array}
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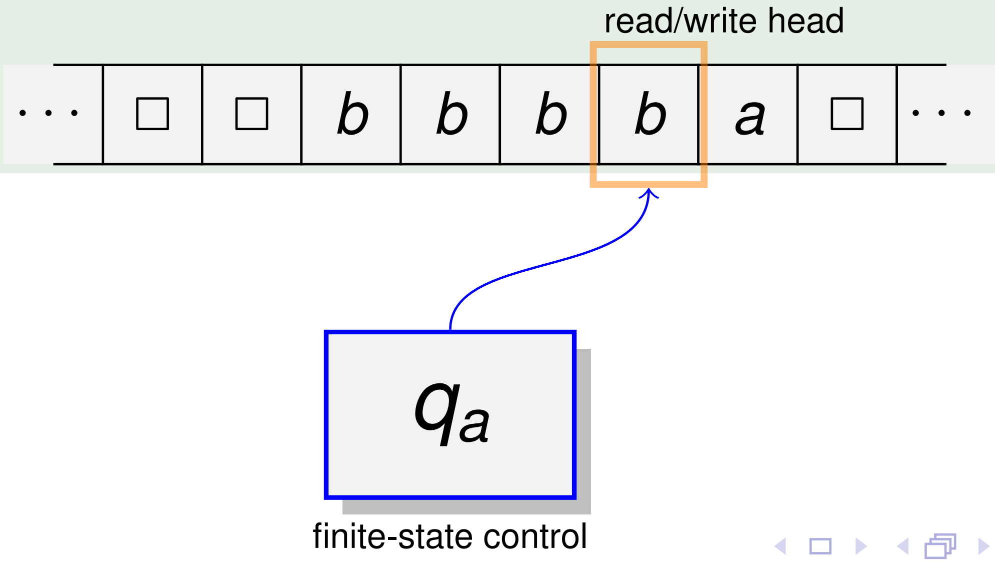
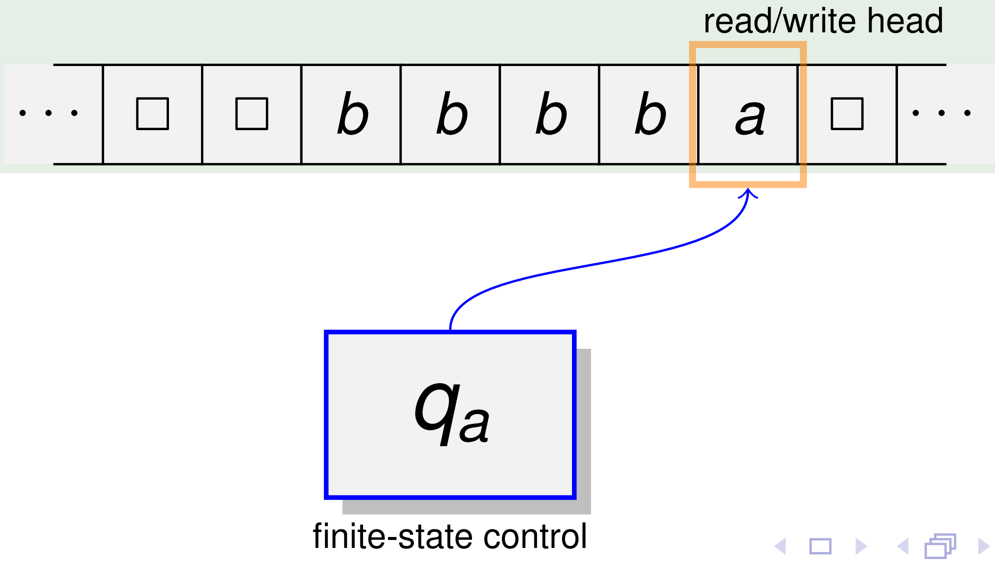


Example (cont.):

The transition function δ is defined as follows:

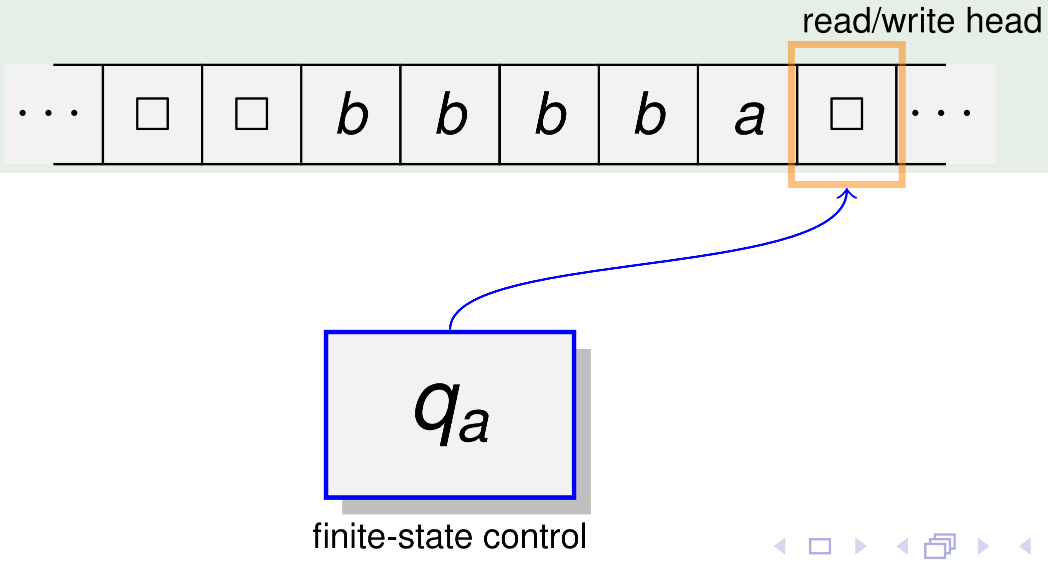
$q_0\Box$	$\to$	$q_1\,a\,0$	$q'_a\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\,\Box\,R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\,\Box\,R$
$q_0 b$	$\to$	$q_b\,\Box\,R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\,\Box\,L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q'_b\,\Box\,L$	$q_3 b$	$\to$	$q_3\,\Box\,L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\,\Box\,L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q'_a\,\Box\,L$	$q'_b b$	$\to$	$q_2\,\Box\,L$			
$q'_a a$	$\to$	$q_2\,\Box\,L$	$q'_b\Box$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\,\Box\,L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

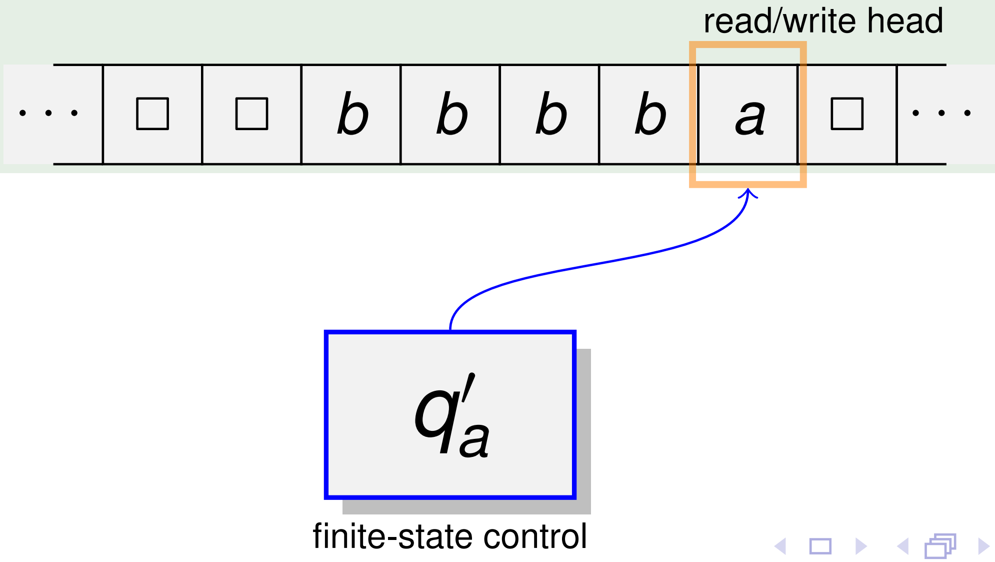
$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
q_0\Box & \to & q_1\,a\,0 & q'_a\Box & \to & q_1\,a\,0 & q_2 b & \to & q_2\,b\,L \\
q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q'_b\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q'_b a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q'_a\Box L & q'_b b & \to & q_2\Box L & & & \\
q'_a a & \to & q_2\Box L & q'_b\Box & \to & q_1\,a\,0 & & & \\
q'_a b & \to & q_3\Box L & q_2 a & \to & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

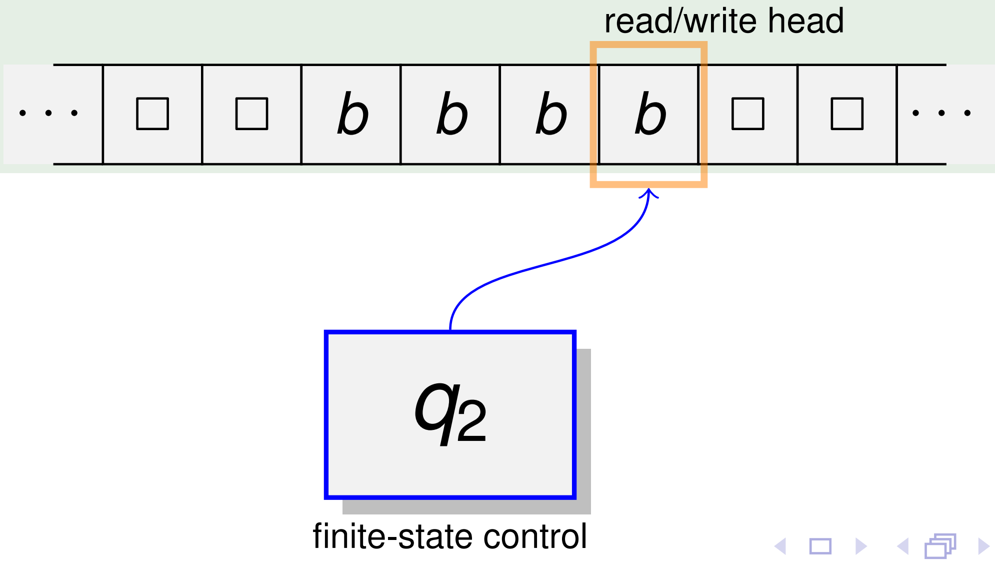
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\Box R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\Box R$
$q_0 b$	$\rightarrow$	$q_b\Box R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\Box L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\Box L$	$q_3 b$	$\rightarrow$	$q_3\Box L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\Box L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\Box L$	$q'_b b$	$\rightarrow$	$q_2\Box L$			
$q'_a a$	$\rightarrow$	$q_2\Box L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\Box L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

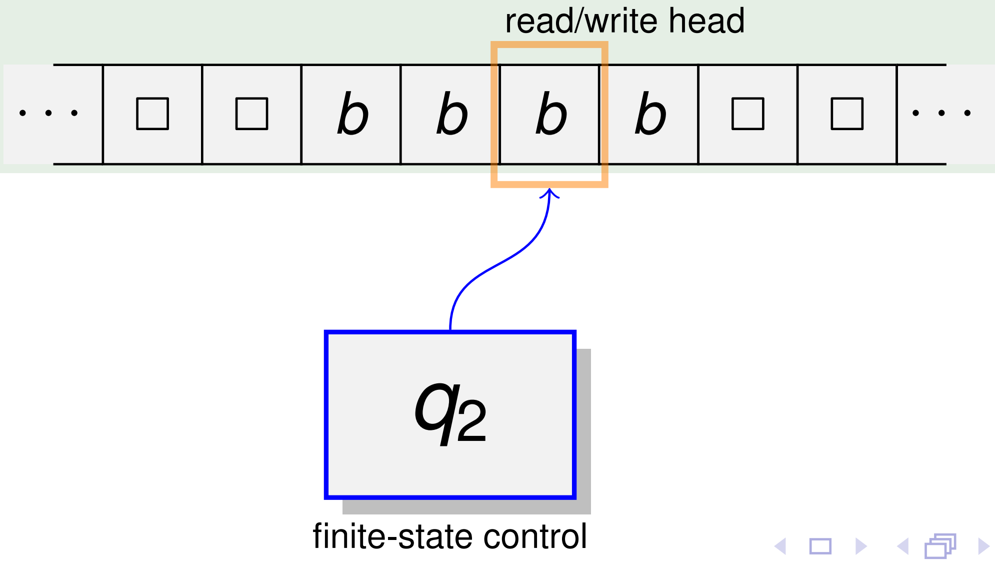
$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
q_0\Box & \to & q_1\,a\,0 & q'_a\Box & \to & q_1\,a\,0 & q_2 b & \to & q_2\,b\,L \\
q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q'_b\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q'_b a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q'_a\Box L & q'_b b & \to & q_2\Box L & & & \\
q'_a a & \to & q_2\Box L & q'_b\Box & \to & q_1\,a\,0 & & & \\
q'_a b & \to & q_3\Box L & q_2 a & \to & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

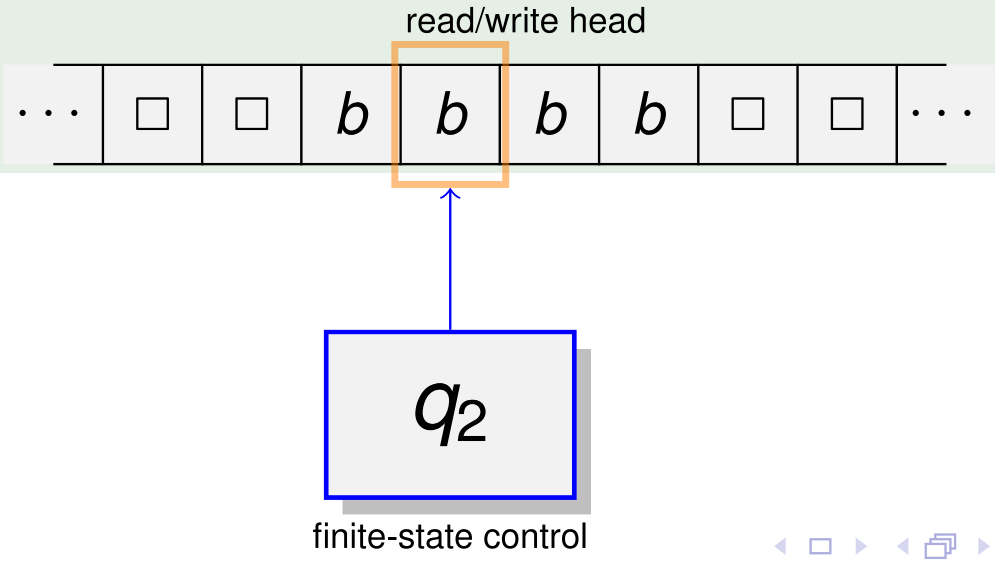
$$
\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
q_0\Box & \to & q_1\,a\,0 & q_a'\Box & \to & q_1\,a\,0 & q_2 b & \to & q_2\,b\,L \\
q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q_b'\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q_b' a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q_a'\Box L & q_b' b & \to & q_2\Box L & & & \\
q_a' a & \to & q_2\Box L & q_b'\Box & \to & q_1\,a\,0 & & & \\
q_a' b & \to & q_3\Box L & q_2 a & \to & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

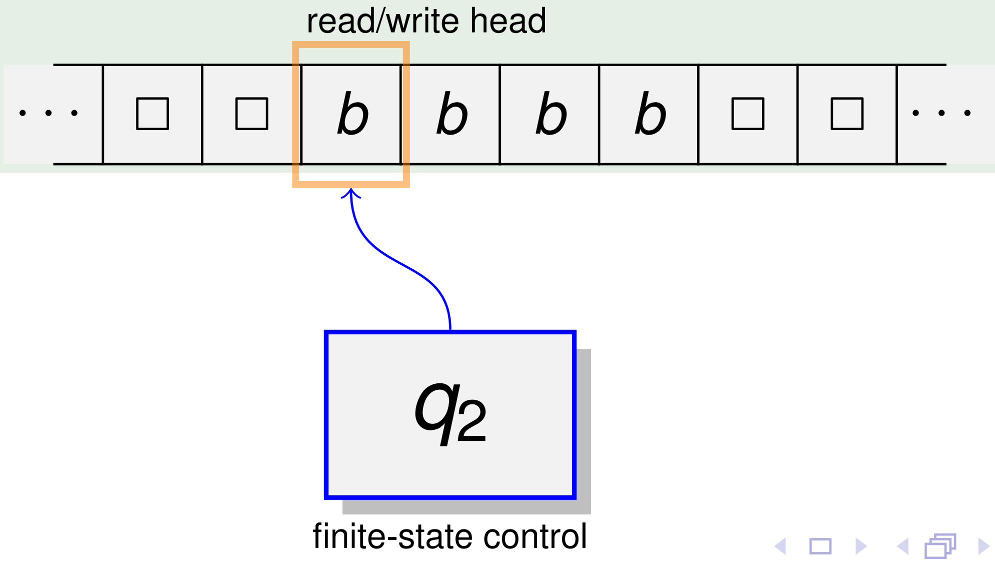
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\Box R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\Box R$
$q_0 b$	$\rightarrow$	$q_b\Box R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\Box L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\Box L$	$q_3 b$	$\rightarrow$	$q_3\Box L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\Box L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\Box L$	$q'_b b$	$\rightarrow$	$q_2\Box L$			
$q'_a a$	$\rightarrow$	$q_2\Box L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\Box L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

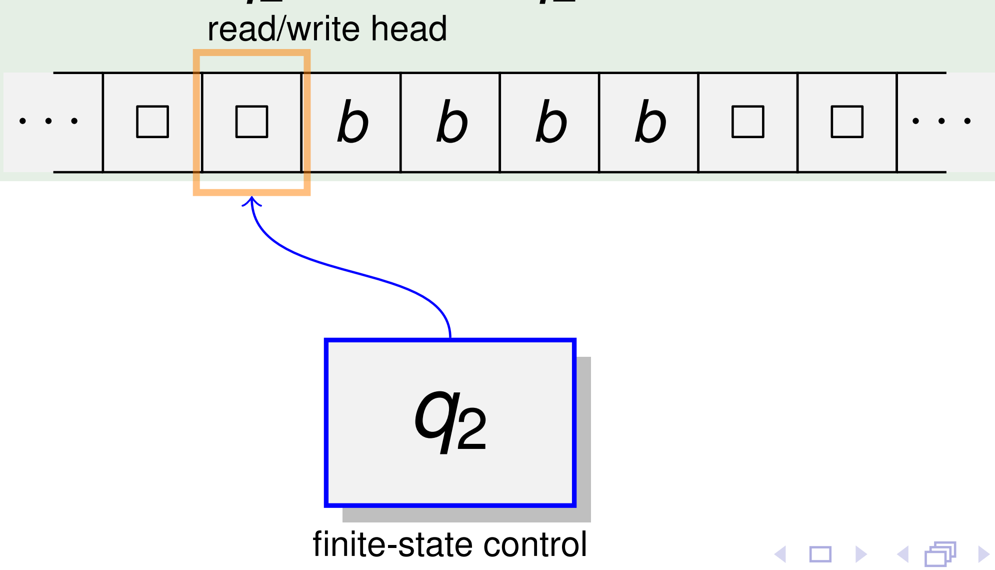
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\Box R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\Box R$
$q_0 b$	$\rightarrow$	$q_b\Box R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\Box L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\Box L$	$q_3 b$	$\rightarrow$	$q_3\Box L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\Box L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\Box L$	$q'_b b$	$\rightarrow$	$q_2\Box L$			
$q'_a a$	$\rightarrow$	$q_2\Box L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\Box L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

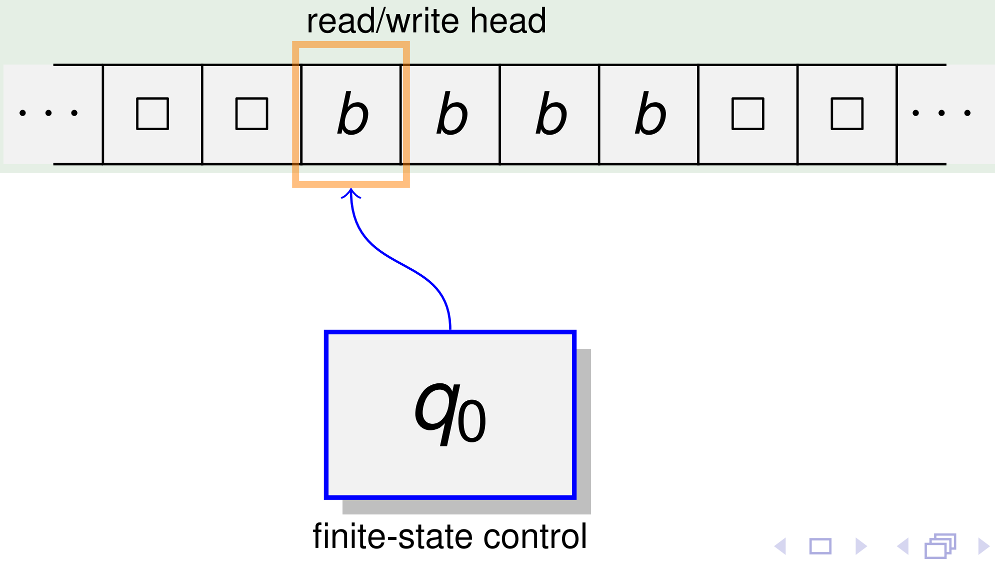
$q_0 \square$	$\to$	$q_1 a 0$	$q'_a \square$	$\to$	$q_1 a 0$	$q_2 b$	$\to$	$q_2 b L$
$q_0 a$	$\to$	$q_a \square R$	$q_b a$	$\to$	$q_b a R$	$q_2 \square$	$\to$	$q_0 \square R$
$q_0 b$	$\to$	$q_b \square R$	$q_b b$	$\to$	$q_b b R$	$q_3 a$	$\to$	$q_3 \square L$
$q_a a$	$\to$	$q_a a R$	$q_b \square$	$\to$	$q'_b \square L$	$q_3 b$	$\to$	$q_3 \square L$
$q_a b$	$\to$	$q_a b R$	$q'_b a$	$\to$	$q_3 \square L$	$q_3 \square$	$\to$	$q_1 b 0$
$q_a \square$	$\to$	$q'_a \square L$	$q'_b b$	$\to$	$q_2 \square L$			
$q'_a a$	$\to$	$q_2 \square L$	$q'_b \square$	$\to$	$q_1 a 0$			
$q'_a b$	$\to$	$q_3 \square L$	$q_2 a$	$\to$	$q_2 a L$			



Example (cont.):

The transition function δ is defined as follows:

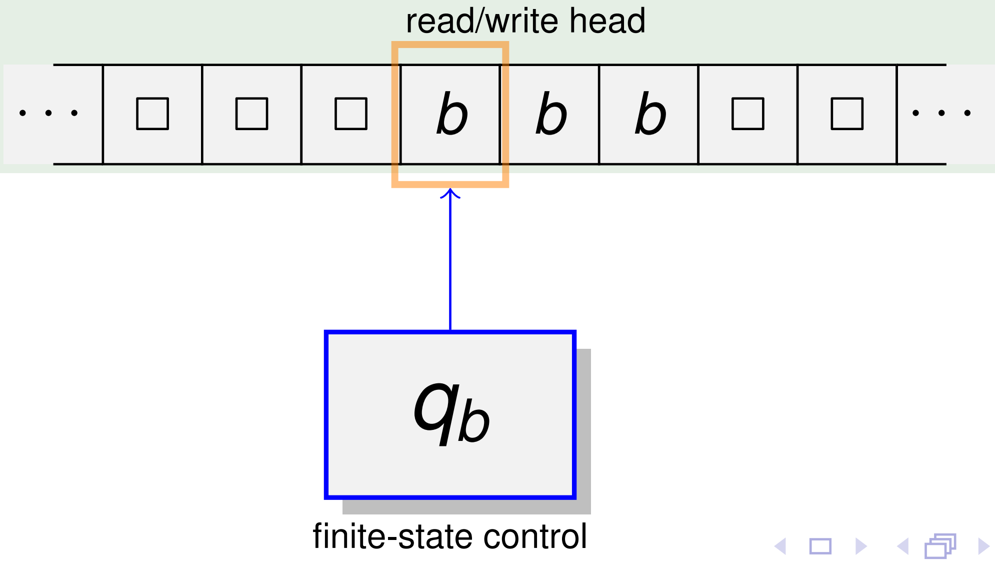
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\Box R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\Box R$
$q_0 b$	$\rightarrow$	$q_b\Box R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\Box L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\Box L$	$q_3 b$	$\rightarrow$	$q_3\Box L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\Box L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\Box L$	$q'_b b$	$\rightarrow$	$q_2\Box L$			
$q'_a a$	$\rightarrow$	$q_2\Box L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\Box L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

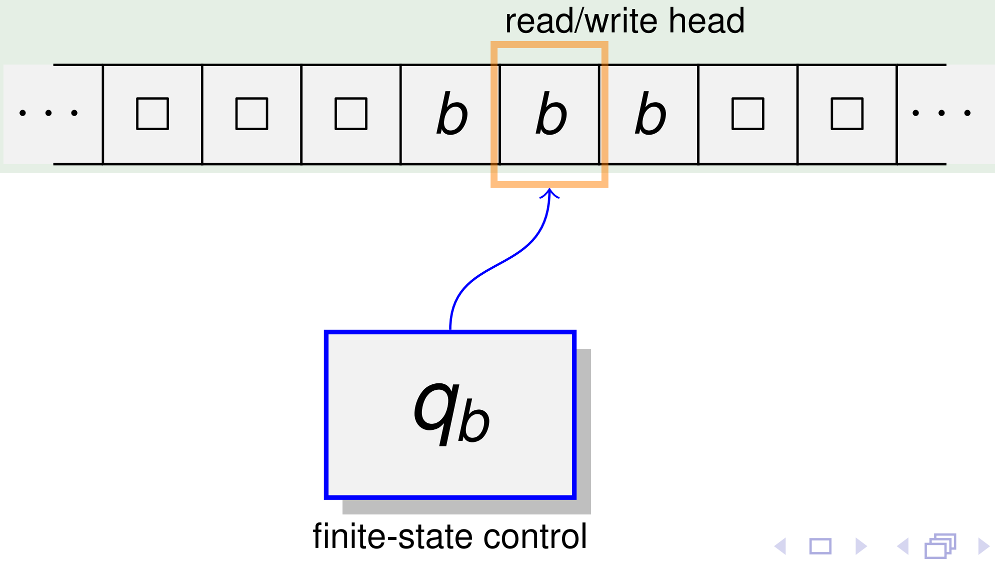
$q_0\Box$	$\to$	$q_1 a 0$	$q'_a\Box$	$\to$	$q_1 a 0$	$q_2 b$	$\to$	$q_2 b L$
$q_0 a$	$\to$	$q_a \Box R$	$q_b a$	$\to$	$q_b a R$	$q_2\Box$	$\to$	$q_0 \Box R$
$q_0 b$	$\to$	$q_b \Box R$	$q_b b$	$\to$	$q_b b R$	$q_3 a$	$\to$	$q_3 \Box L$
$q_a a$	$\to$	$q_a a R$	$q_b\Box$	$\to$	$q'_b \Box L$	$q_3 b$	$\to$	$q_3 \Box L$
$q_a b$	$\to$	$q_a b R$	$q'_b a$	$\to$	$q_3 \Box L$	$q_3\Box$	$\to$	$q_1 b 0$
$q_a\Box$	$\to$	$q'_a \Box L$	$q'_b b$	$\to$	$q_2 \Box L$			
$q'_a a$	$\to$	$q_2 \Box L$	$q'_b\Box$	$\to$	$q_1 a 0$			
$q'_a b$	$\to$	$q_3 \Box L$	$q_2 a$	$\to$	$q_2 a L$			



Example (cont.):

The transition function δ is defined as follows:

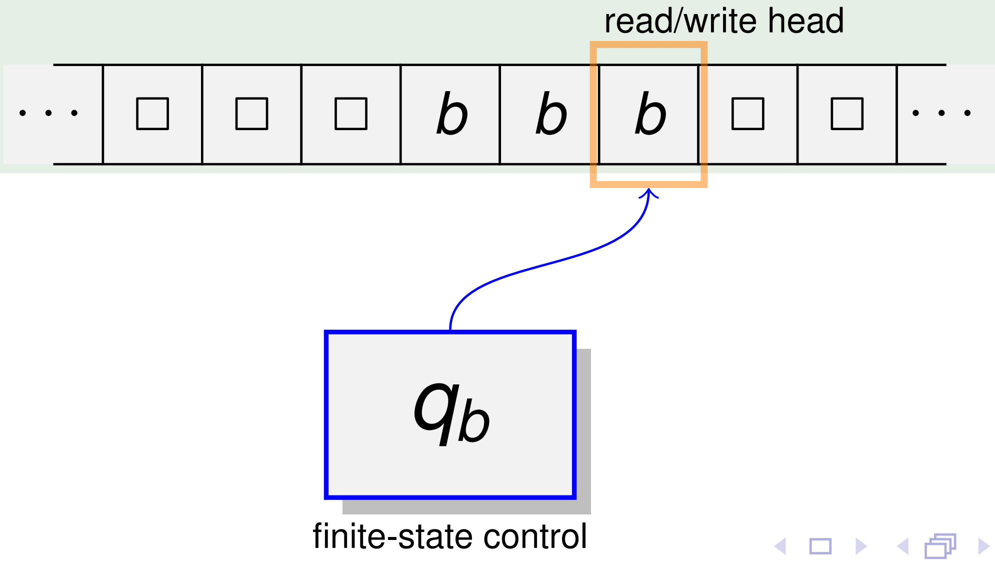
$q_0\Box$	$\to$	$q_1\,a\,0$	$q'_a\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\Box R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\Box R$
$q_0 b$	$\to$	$q_b\Box R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\Box L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q'_b\Box L$	$q_3 b$	$\to$	$q_3\Box L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\Box L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q'_a\Box L$	$q'_b b$	$\to$	$q_2\Box L$			
$q'_a a$	$\to$	$q_2\Box L$	$q'_b\Box$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\Box L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

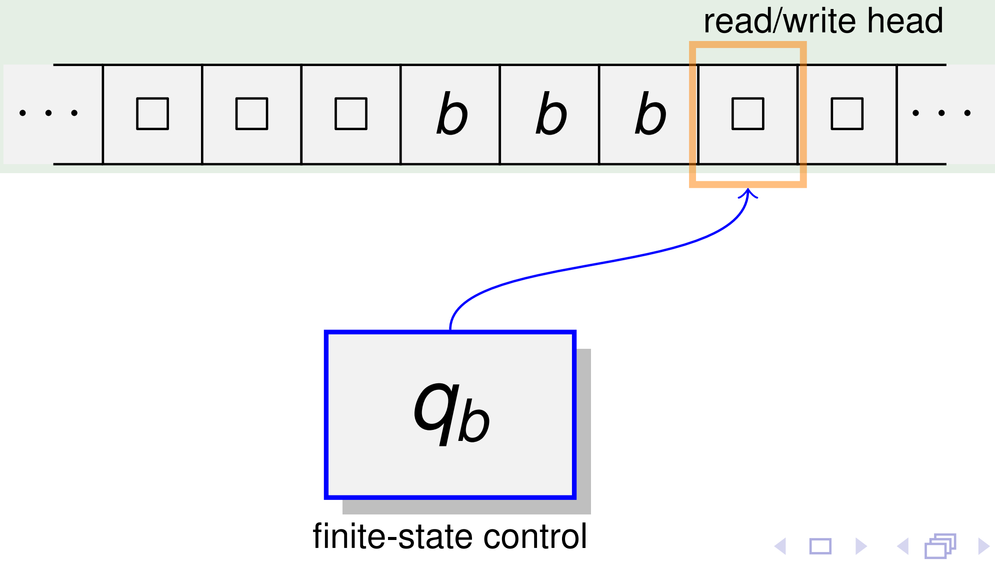
$q_0\Box$	$\to$	$q_1\,a\,0$	$q_a'\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\,\Box\,R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\,\Box\,R$
$q_0 b$	$\to$	$q_b\,\Box\,R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\,\Box\,L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q_b'\,\Box\,L$	$q_3 b$	$\to$	$q_3\,\Box\,L$
$q_a b$	$\to$	$q_a\,b\,R$	$q_b' a$	$\to$	$q_3\,\Box\,L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q_a'\,\Box\,L$	$q_b' b$	$\to$	$q_2\,\Box\,L$			
$q_a' a$	$\to$	$q_2\,\Box\,L$	$q_b'\Box$	$\to$	$q_1\,a\,0$			
$q_a' b$	$\to$	$q_3\,\Box\,L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

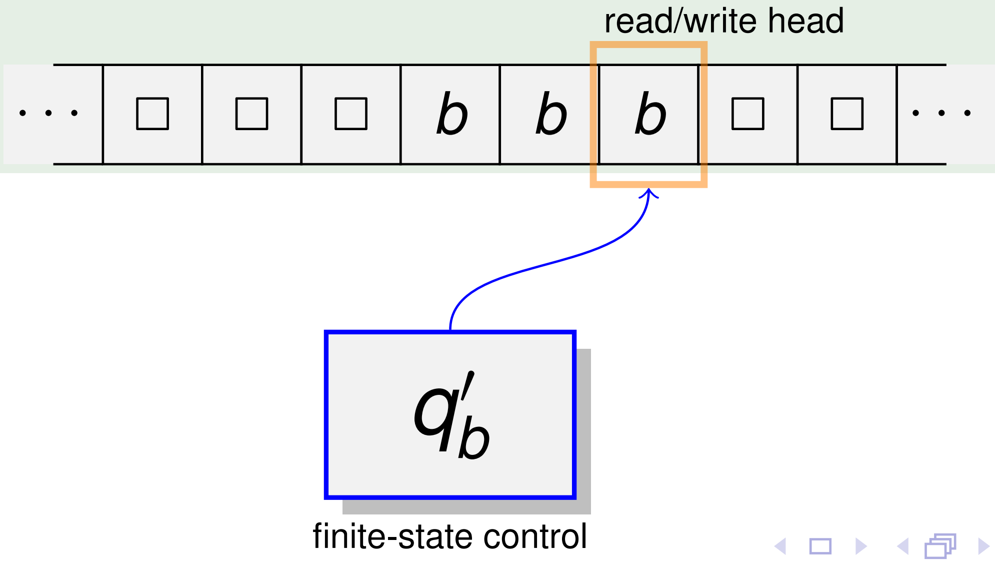
$$
\begin{array}{lll@{\qquad}lll@{\qquad}lll}
q_0\Box & \rightarrow & q_1\,a\,0 & q'_a\Box & \rightarrow & q_1\,a\,0 & q_2 b & \rightarrow & q_2\,b\,L \\
q_0 a & \rightarrow & q_a\Box R & q_b a & \rightarrow & q_b\,a\,R & q_2\Box & \rightarrow & q_0\Box R \\
q_0 b & \rightarrow & q_b\Box R & q_b b & \rightarrow & q_b\,b\,R & q_3 a & \rightarrow & q_3\Box L \\
q_a a & \rightarrow & q_a\,a\,R & q_b\Box & \rightarrow & q'_b\Box L & q_3 b & \rightarrow & q_3\Box L \\
q_a b & \rightarrow & q_a\,b\,R & q'_b a & \rightarrow & q_3\Box L & q_3\Box & \rightarrow & q_1\,b\,0 \\
q_a\Box & \rightarrow & q'_a\Box L & q'_b b & \rightarrow & q_2\Box L & & & \\
q'_a a & \rightarrow & q_2\Box L & q'_b\Box & \rightarrow & q_1\,a\,0 & & & \\
q'_a b & \rightarrow & q_3\Box L & q_2 a & \rightarrow & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

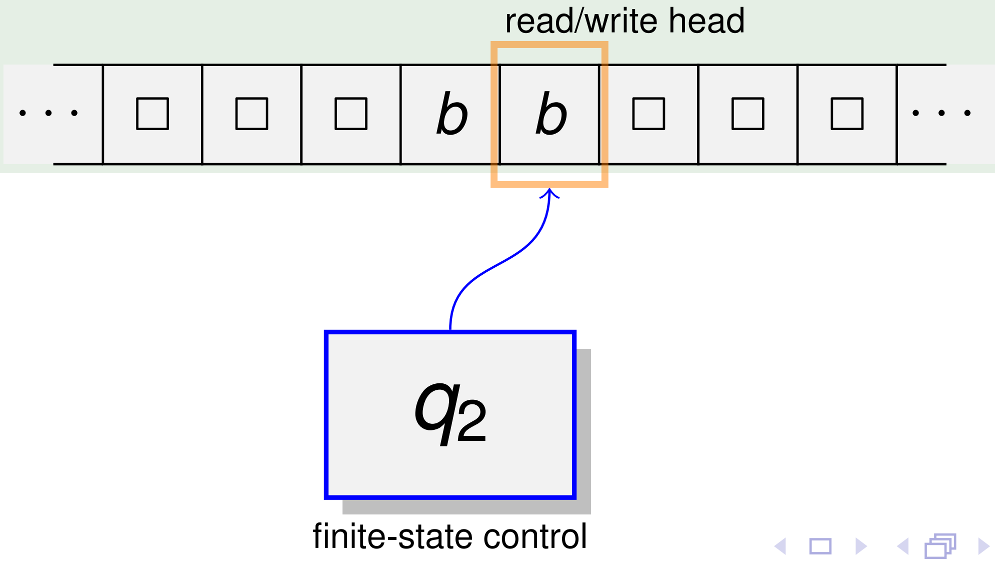
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\Box R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\Box R$
$q_0 b$	$\rightarrow$	$q_b\Box R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\Box L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\Box L$	$q_3 b$	$\rightarrow$	$q_3\Box L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\Box L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\Box L$	$q'_b b$	$\rightarrow$	$q_2\Box L$			
$q'_a a$	$\rightarrow$	$q_2\Box L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\Box L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

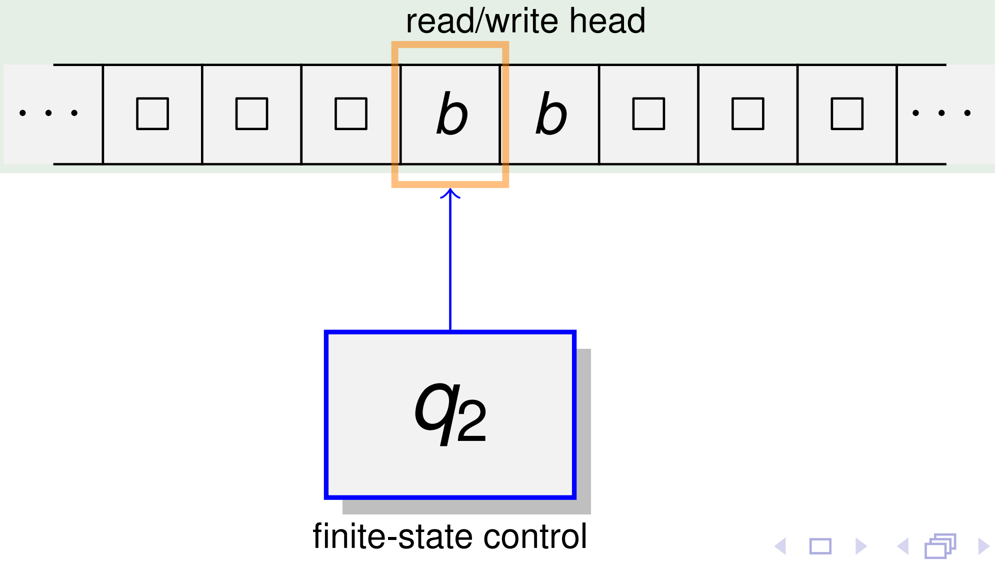
$q_0\Box$	$\to$	$q_1\,a\,0$	$q'_a\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\Box R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\Box R$
$q_0 b$	$\to$	$q_b\Box R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\Box L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q'_b\Box L$	$q_3 b$	$\to$	$q_3\Box L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\Box L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q'_a\Box L$	$q'_b b$	$\to$	$q_2\Box L$			
$q'_a a$	$\to$	$q_2\Box L$	$q'_b\Box$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\Box L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

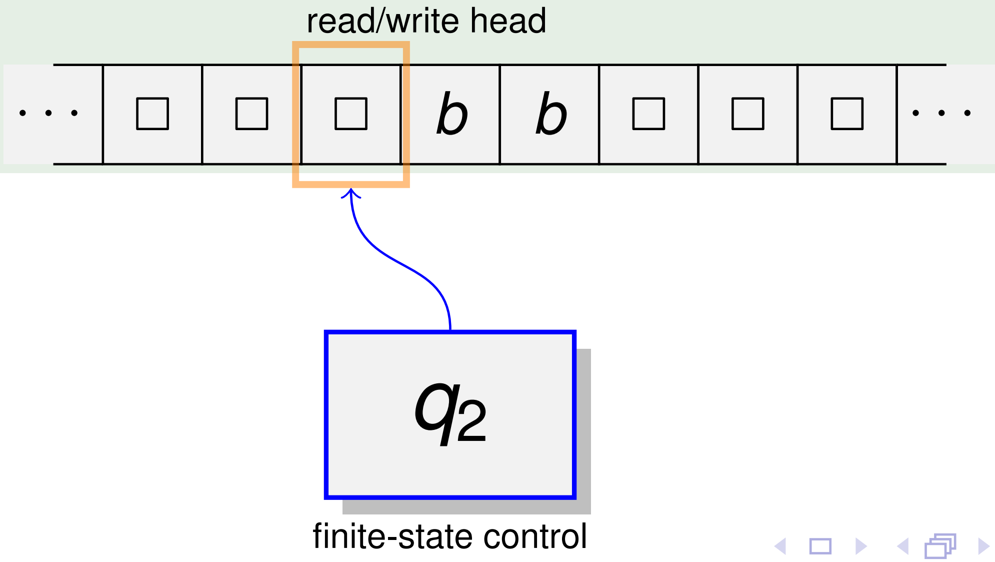
$q_0\Box$	$\rightarrow$	$q_1\,a\,0$	$q'_a\Box$	$\rightarrow$	$q_1\,a\,0$	$q_2 b$	$\rightarrow$	$q_2\,b\,L$
$q_0 a$	$\rightarrow$	$q_a\,\Box\,R$	$q_b a$	$\rightarrow$	$q_b\,a\,R$	$q_2\Box$	$\rightarrow$	$q_0\,\Box\,R$
$q_0 b$	$\rightarrow$	$q_b\,\Box\,R$	$q_b b$	$\rightarrow$	$q_b\,b\,R$	$q_3 a$	$\rightarrow$	$q_3\,\Box\,L$
$q_a a$	$\rightarrow$	$q_a\,a\,R$	$q_b\Box$	$\rightarrow$	$q'_b\,\Box\,L$	$q_3 b$	$\rightarrow$	$q_3\,\Box\,L$
$q_a b$	$\rightarrow$	$q_a\,b\,R$	$q'_b a$	$\rightarrow$	$q_3\,\Box\,L$	$q_3\Box$	$\rightarrow$	$q_1\,b\,0$
$q_a\Box$	$\rightarrow$	$q'_a\,\Box\,L$	$q'_b b$	$\rightarrow$	$q_2\,\Box\,L$			
$q'_a a$	$\rightarrow$	$q_2\,\Box\,L$	$q'_b\Box$	$\rightarrow$	$q_1\,a\,0$			
$q'_a b$	$\rightarrow$	$q_3\,\Box\,L$	$q_2 a$	$\rightarrow$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

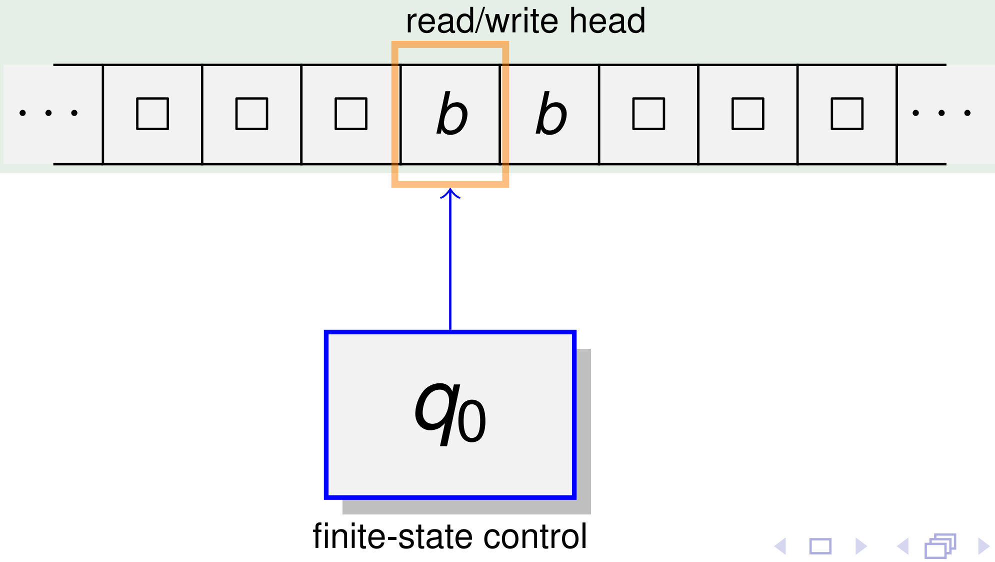
$q_0\square$	$\to$	$q_1\,a\,0$	$q'_a\square$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\square R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\square$	$\to$	$q_0\square R$
$q_0 b$	$\to$	$q_b\square R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\square L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\square$	$\to$	$q'_b\square L$	$q_3 b$	$\to$	$q_3\square L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\square L$	$q_3\square$	$\to$	$q_1\,b\,0$
$q_a\square$	$\to$	$q'_a\square L$	$q'_b b$	$\to$	$q_2\square L$			
$q'_a a$	$\to$	$q_2\square L$	$q'_b\square$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\square L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

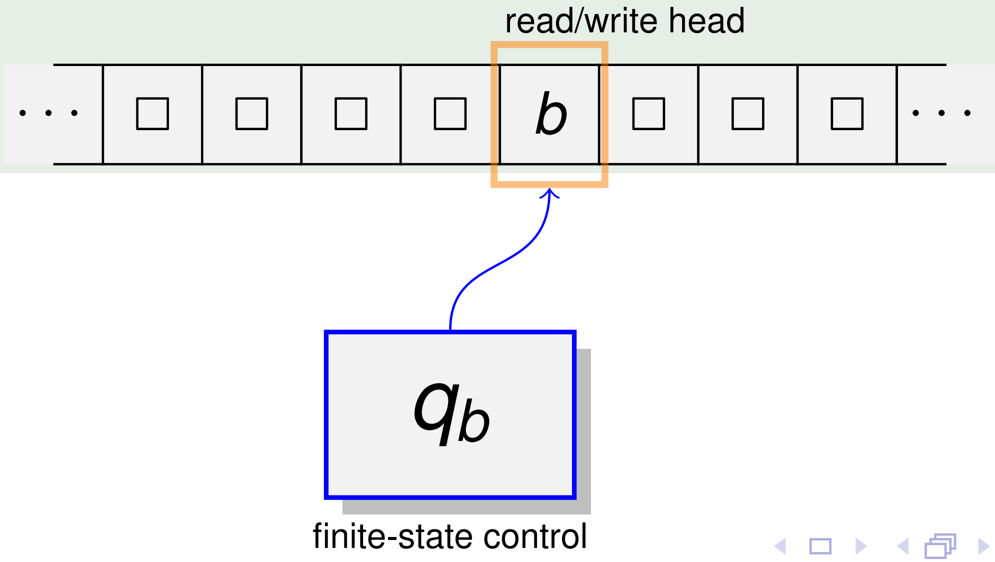
$q_0\Box$	$\to$	$q_1\,a\,0$	$q'_a\Box$	$\to$	$q_1\,a\,0$	$q_2 b$	$\to$	$q_2\,b\,L$
$q_0 a$	$\to$	$q_a\,\Box\,R$	$q_b a$	$\to$	$q_b\,a\,R$	$q_2\Box$	$\to$	$q_0\,\Box\,R$
$q_0 b$	$\to$	$q_b\,\Box\,R$	$q_b b$	$\to$	$q_b\,b\,R$	$q_3 a$	$\to$	$q_3\,\Box\,L$
$q_a a$	$\to$	$q_a\,a\,R$	$q_b\Box$	$\to$	$q'_b\,\Box\,L$	$q_3 b$	$\to$	$q_3\,\Box\,L$
$q_a b$	$\to$	$q_a\,b\,R$	$q'_b a$	$\to$	$q_3\,\Box\,L$	$q_3\Box$	$\to$	$q_1\,b\,0$
$q_a\Box$	$\to$	$q'_a\,\Box\,L$	$q'_b b$	$\to$	$q_2\,\Box\,L$			
$q'_a a$	$\to$	$q_2\,\Box\,L$	$q'_b\Box$	$\to$	$q_1\,a\,0$			
$q'_a b$	$\to$	$q_3\,\Box\,L$	$q_2 a$	$\to$	$q_2\,a\,L$			



Example (cont.):

The transition function δ is defined as follows:

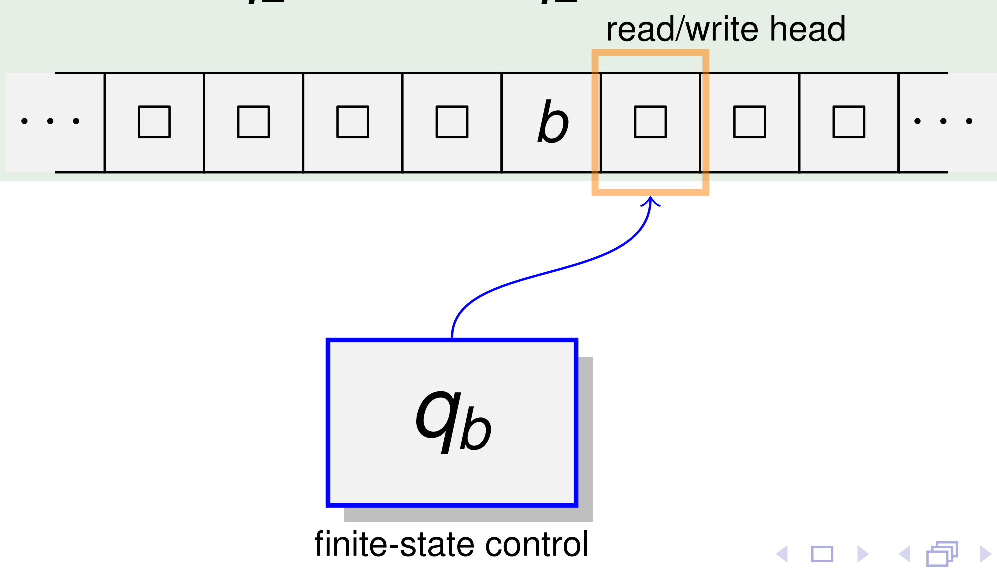
$q_0\Box \to q_1 a 0$	$q'_a\Box \to q_1 a 0$	$q_2 b \to q_2 b L$
$q_0 a \to q_a \Box R$	$q_b a \to q_b a R$	$q_2\Box \to q_0 \Box R$
$q_0 b \to q_b \Box R$	$q_b b \to q_b b R$	$q_3 a \to q_3 \Box L$
$q_a a \to q_a a R$	$q_b\Box \to q'_b \Box L$	$q_3 b \to q_3 \Box L$
$q_a b \to q_a b R$	$q'_b a \to q_3 \Box L$	$q_3\Box \to q_1 b 0$
$q_a\Box \to q'_a \Box L$	$q'_b b \to q_2 \Box L$	
$q'_a a \to q_2 \Box L$	$q'_b\Box \to q_1 a 0$	
$q'_a b \to q_3 \Box L$	$q_2 a \to q_2 a L$	



Example (cont.):

The transition function δ is defined as follows:

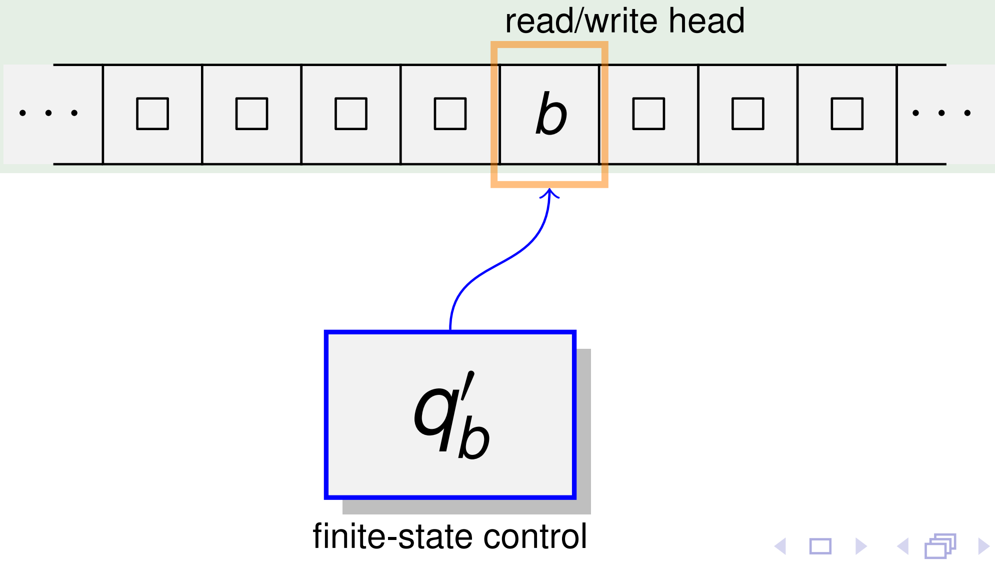
$$
\begin{array}{lllllllll}
q_0\Box & \to & q_1\,a\,0 & q'_a\Box & \to & q_1\,a\,0 & q_2 b & \to & q_2\,b\,L \\
q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q'_b\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q'_b a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q'_a\Box L & q'_b b & \to & q_2\Box L & & & \\
q'_a a & \to & q_2\Box L & q'_b\Box & \to & q_1\,a\,0 & & & \\
q'_a b & \to & q_3\Box L & q_2 a & \to & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

$q_0 \Box$	$\rightarrow$	$q_1\, a\, 0$	$q'_a \Box$	$\rightarrow$	$q_1\, a\, 0$	$q_2\, b$	$\rightarrow$	$q_2\, b\, L$
$q_0\, a$	$\rightarrow$	$q_a \Box R$	$q_b\, a$	$\rightarrow$	$q_b\, a\, R$	$q_2 \Box$	$\rightarrow$	$q_0 \Box R$
$q_0\, b$	$\rightarrow$	$q_b \Box R$	$q_b\, b$	$\rightarrow$	$q_b\, b\, R$	$q_3\, a$	$\rightarrow$	$q_3 \Box L$
$q_a\, a$	$\rightarrow$	$q_a\, a\, R$	$q_b \Box$	$\rightarrow$	$q'_b \Box L$	$q_3\, b$	$\rightarrow$	$q_3 \Box L$
$q_a\, b$	$\rightarrow$	$q_a\, b\, R$	$q'_b\, a$	$\rightarrow$	$q_3 \Box L$	$q_3 \Box$	$\rightarrow$	$q_1\, b\, 0$
$q_a \Box$	$\rightarrow$	$q'_a \Box L$	$q'_b\, b$	$\rightarrow$	$q_2 \Box L$			
$q'_a\, a$	$\rightarrow$	$q_2 \Box L$	$q'_b \Box$	$\rightarrow$	$q_1\, a\, 0$			
$q'_a\, b$	$\rightarrow$	$q_3 \Box L$	$q_2\, a$	$\rightarrow$	$q_2\, a\, L$			



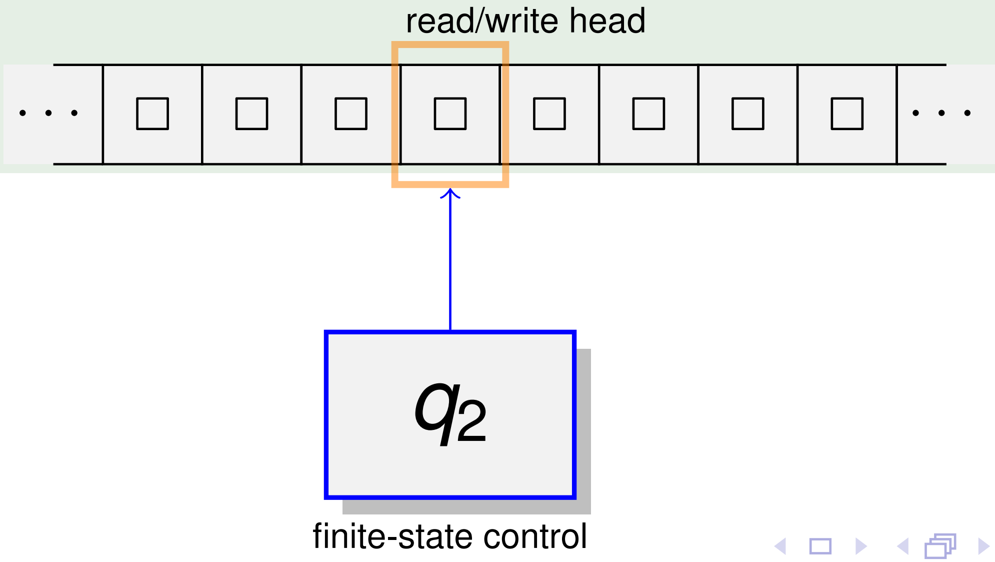
Example (cont.):

The transition function δ is defined as follows:

$$
\begin{array}{lcl}
q_0\Box & \rightarrow & q_1\,a\,0 \\
q_0 a & \rightarrow & q_a\Box R \\
q_0 b & \rightarrow & q_b\Box R \\
q_a a & \rightarrow & q_a a\,R \\
q_a b & \rightarrow & q_a b\,R \\
q_a\Box & \rightarrow & q'_a\Box L \\
q'_a a & \rightarrow & q_2\Box L \\
q'_a b & \rightarrow & q_3\Box L
\end{array}
$$

$$
\begin{array}{lcl}
q'_a\Box & \rightarrow & q_1\,a\,0 \\
q_b a & \rightarrow & q_b a\,R \\
q_b b & \rightarrow & q_b b\,R \\
q_b\Box & \rightarrow & q'_b\Box L \\
q'_b a & \rightarrow & q_3\Box L \\
q'_b b & \rightarrow & q_2\Box L \\
q'_b\Box & \rightarrow & q_1\,a\,0 \\
q_2 a & \rightarrow & q_2 a\,L
\end{array}
$$

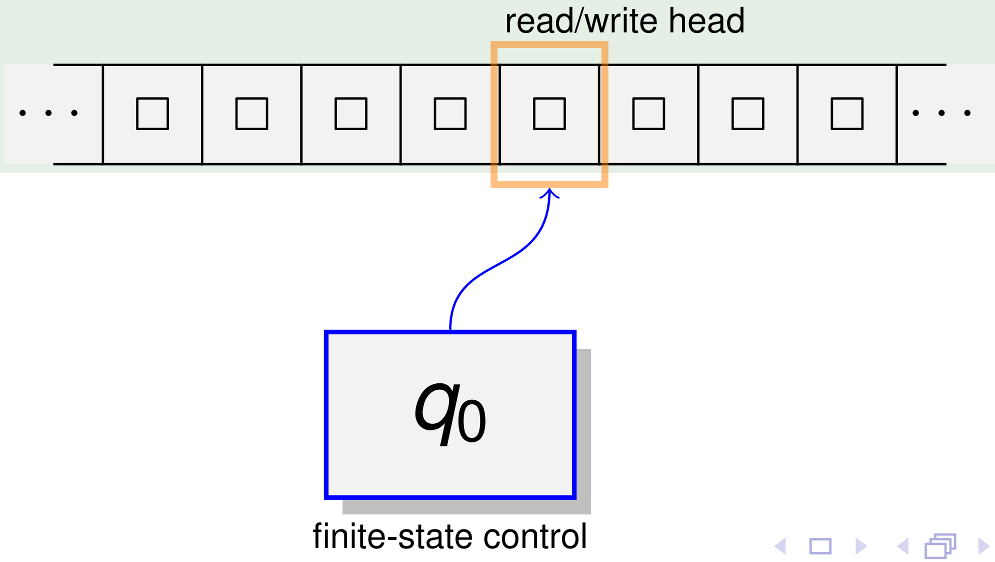
$$
\begin{array}{lcl}
q_2 b & \rightarrow & q_2 b\,L \\
q_2\Box & \rightarrow & q_0\Box R \\
q_3 a & \rightarrow & q_3\Box L \\
q_3 b & \rightarrow & q_3\Box L \\
q_3\Box & \rightarrow & q_1\,b\,0
\end{array}
$$



Example (cont.):

The transition function δ is defined as follows:

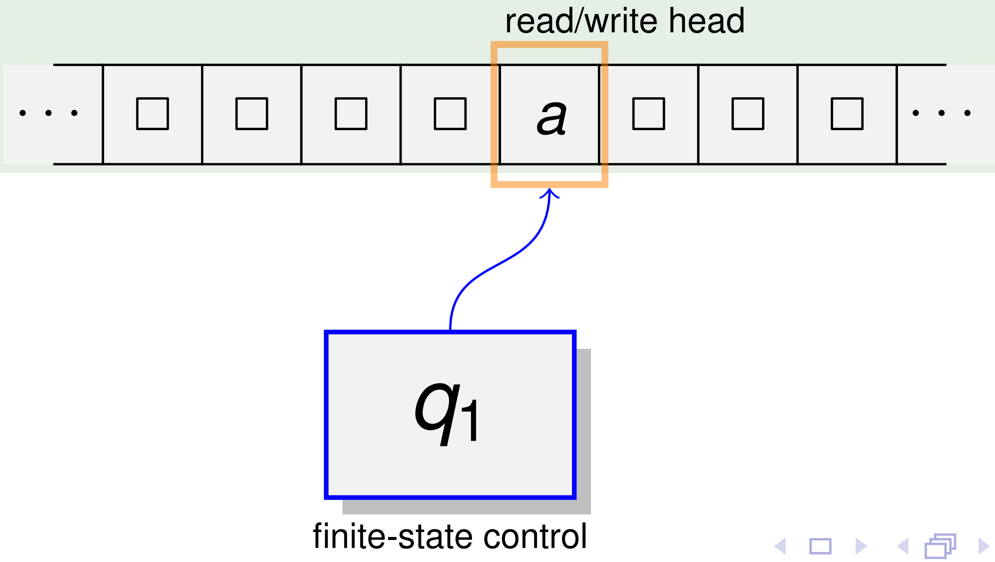
$$
\begin{array}{lcl@{\qquad}lcl@{\qquad}lcl}
q_0\Box & \to & q_1\,a\,0 & q'_a\Box & \to & q_1\,a\,0 & q_2 b & \to & q_2\,b\,L \\
q_0 a & \to & q_a\Box R & q_b a & \to & q_b\,a\,R & q_2\Box & \to & q_0\Box R \\
q_0 b & \to & q_b\Box R & q_b b & \to & q_b\,b\,R & q_3 a & \to & q_3\Box L \\
q_a a & \to & q_a\,a\,R & q_b\Box & \to & q'_b\Box L & q_3 b & \to & q_3\Box L \\
q_a b & \to & q_a\,b\,R & q'_b a & \to & q_3\Box L & q_3\Box & \to & q_1\,b\,0 \\
q_a\Box & \to & q'_a\Box L & q'_b b & \to & q_2\Box L & & & \\
q'_a a & \to & q_2\Box L & q'_b\Box & \to & q_1\,a\,0 & & & \\
q'_a b & \to & q_3\Box L & q_2 a & \to & q_2\,a\,L & & &
\end{array}
$$



Example (cont.):

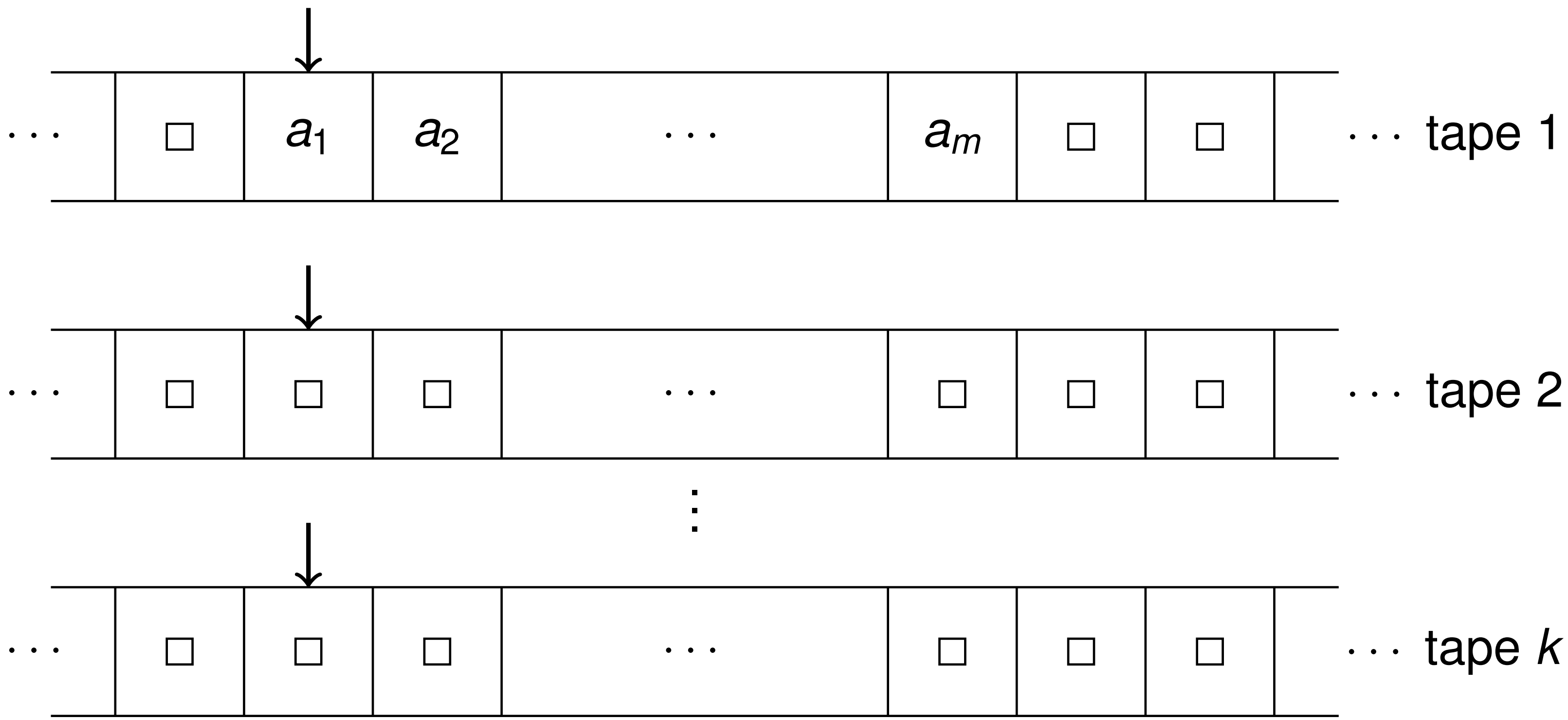
The transition function δ is defined as follows:

$q_0\square$	$\rightarrow$	$q_1 a 0$	$q_a'\square$	$\rightarrow$	$q_1 a 0$	$q_2 b$	$\rightarrow$	$q_2 b L$
$q_0 a$	$\rightarrow$	$q_a \square R$	$q_b a$	$\rightarrow$	$q_b a R$	$q_2\square$	$\rightarrow$	$q_0 \square R$
$q_0 b$	$\rightarrow$	$q_b \square R$	$q_b b$	$\rightarrow$	$q_b b R$	$q_3 a$	$\rightarrow$	$q_3 \square L$
$q_a a$	$\rightarrow$	$q_a a R$	$q_b\square$	$\rightarrow$	$q_b' \square L$	$q_3 b$	$\rightarrow$	$q_3 \square L$
$q_a b$	$\rightarrow$	$q_a b R$	$q_b' a$	$\rightarrow$	$q_3 \square L$	$q_3\square$	$\rightarrow$	$q_1 b 0$
$q_a\square$	$\rightarrow$	$q_a' \square L$	$q_b' b$	$\rightarrow$	$q_2 \square L$			
$q_a' a$	$\rightarrow$	$q_2 \square L$	$q_b'\square$	$\rightarrow$	$q_1 a 0$			
$q_a' b$	$\rightarrow$	$q_3 \square L$	$q_2 a$	$\rightarrow$	$q_2 a L$			



Let $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$ be a k-tape TM, let $q \in Q$, and let $a_1, a_2, \ldots, a_m \in \Sigma$ be input symbols.

By $S(q, a_1 a_2 \cdots a_m)$ we denote the configuration of M, in which M is in state q and in which the contents of the tapes and the head positions are as follows:



Definition 4.1

(a) *Let* $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$ *be a TM,* $\Sigma_1 \subseteq \Sigma \smallsetminus \{*\}$*,
where* $*$ *is a symbol from* Σ*,* $\Sigma_2 \subseteq \Sigma$*, and* $n \geq 0$*.*

A function $f : (\Sigma_1^*)^n \rightsquigarrow \Sigma_2^*$ *is* ***computed by M****, if the following holds for all* $x_1, x_2, \ldots, x_n \in \Sigma_1^*$*:*

(i) *If* $f(x_1, x_2, \ldots, x_n)$ *is defined, then starting from the configuration* $S(q_0, x_1 * x_2 * \cdots * x_n)$*, M will eventually reach the configuration* $S(q_1, f(x_1, x_2, \ldots, x_n))$*.*

(ii) *If* $f(x_1, x_2, \ldots, x_n)$ *is undefined, then starting from the configuration* $S(q_0, x_1 * x_2 * \cdots * x_n)$*, M will not halt at all, or it will halt in a configuration that is not of the form* $S(q_1, y)$ *for any* $y \in \Sigma^*$*.*

(b) *A function* $f : (\Sigma_1^*)^n \rightsquigarrow \Sigma_2^*$ *is called* ***(Turing) computable****, if there exists a TM that computes f.*

Definition 4.1 (cont.)

(c) *A function $f : \mathbb{N}^n \rightsquigarrow \mathbb{N}$ is called (Turing) computable, if the following dyadic encoding $\psi : (\{1,2\}^*)^n \rightsquigarrow \{1,2\}^*$ is (Turing) computable:*

$$\psi(x_1, x_2, \ldots, x_n) := \mathrm{dya}(f(\mathrm{dya}^{-1}(x_1), \mathrm{dya}^{-1}(x_2), \ldots, \mathrm{dya}^{-1}(x_n))).$$

(d) $\mathbb{TM}$ *denotes the class of functions that are (Turing) computable.*

Instead of the dyadic encoding one could choose an r-adic encoding for any $r > 2$. As the functions

$$\text{dya} \circ \text{ad}_r^{-1} : \{1, 2, \ldots, r\}^* \to \{1, 2\}^*$$

and

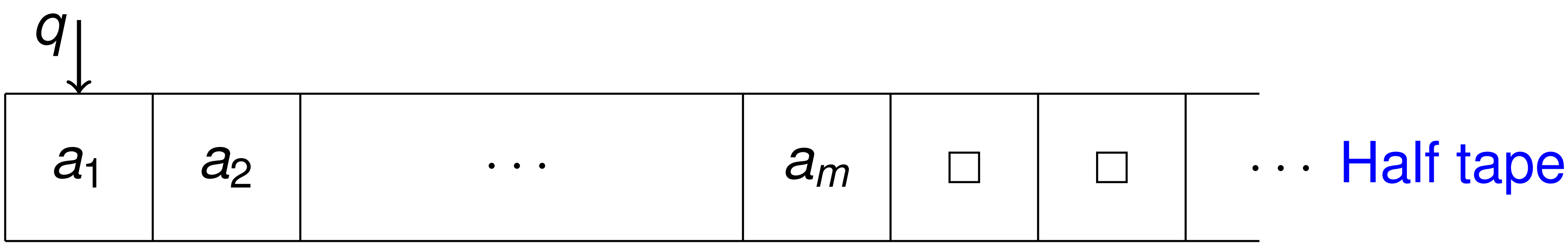
$$\text{ad}_r \circ \text{dya}^{-1} : \{1, 2\}^* \to \{1, 2, \ldots, r\}^*$$

are (Turing) computable, the same class $\mathbb{TM}$ would be obtained.

Example (cont.):

The function $S(x) := x + 1$ is (Turing) computable.

A 1-Turingmachine (1-TM) is a TM with a single tape that is unbounded to the right only:



A transition $qa_1 \to q'bL$ has the same effect as $qa_1 \to 1'b\,0$.

The global situation above is denoted by $S(q, a_1a_2\cdots a_m)$.

Using 1-TMs, we obtain the class 1-$\mathbb{TM}$ of functions on $\mathbb{N}$ that are computable by 1-TMs.

Lemma 4.2

Let f be an n-ary function that is (1-)Turing computable. Then there exists a (1-)TM $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$ *that computes f and that satisfies the following condition:*

$\forall x_1, x_2, \ldots, x_n \in \Sigma_1^*$ *: M halts eventually starting from the configuration* $S(q_0, x_1 * x_2 * \cdots * x_n)$ *if and only if* $f(x_1, x_2, \ldots, x_n)$ *is defined.*

Theorem 4.3

$\mathbb{TM} \subseteq 1\text{-}\mathbb{TM}$.

Proof of Theorem 4.3.

Let $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$ be a TM with k tapes that computes the function $f : (\Sigma_1^*)^n \rightsquigarrow \Sigma_2^*$, where $\Sigma_1 \subseteq \Sigma \setminus \{*\}$ and $\Sigma_2 \subseteq \Sigma$.

By Lemma 4.2 we can assume that, for all $x_1, x_2, \ldots, x_n \in \Sigma_1^*$, M halts eventually starting from $S(q_0, x_1 * x_2 * \cdots * x_n)$ iff $f(x_1, x_2, \ldots, x_n)$ is defined.

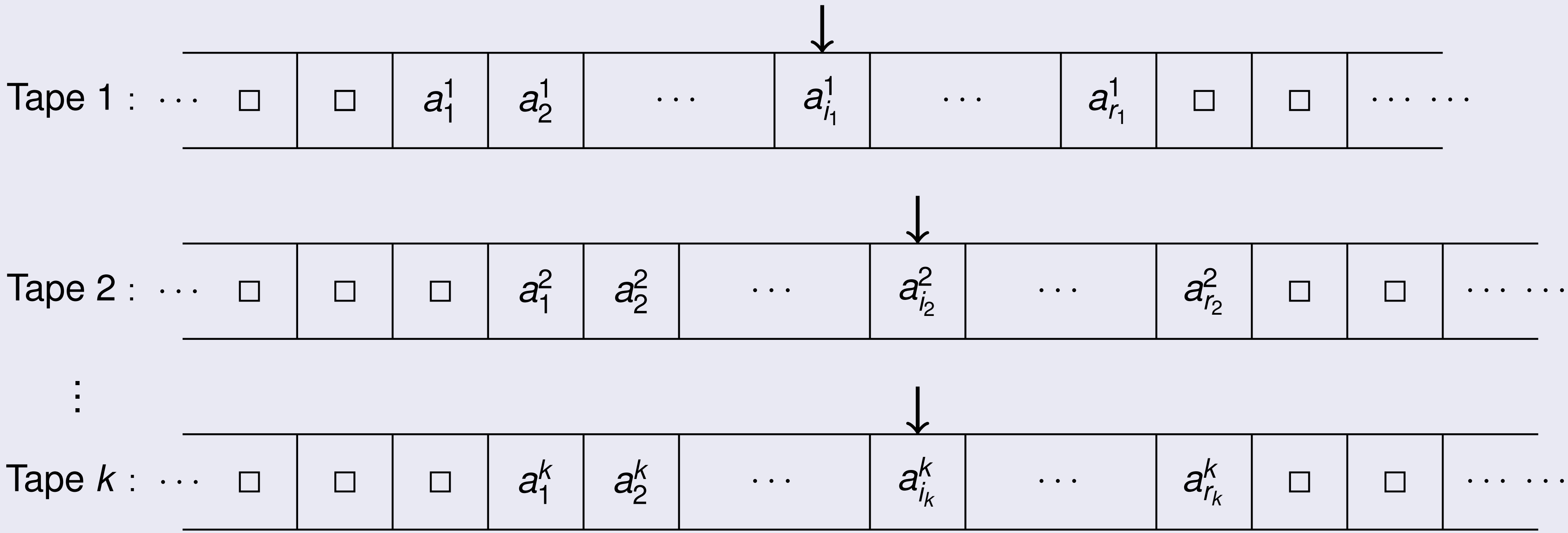
We describe a 1-TM $M' = (Q', \Sigma, \Gamma', \Box, \delta', q_0', q_1')$ that computes the function f by simulating M.

The alphabet Σ' includes Σ together with the following symbols: $\blacksquare, \nabla, \blacklozenge$.

On the tape of M', each configuration of M is encoded in a special way.

Proof of Theorem 4.3 (cont.)

Assume that M is in the configuration that is given by the state $q \in Q$ and the following tape situation:



Proof of Theorem 4.3 (cont.)

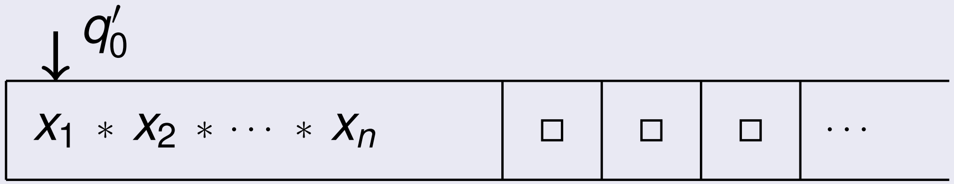
This configuration is encoded by the configuration of M' that is given by the state $(q, 1) \in Q'$ and the following tape situation:

↓																														
■	a_1^1	a_2^1	$\cdots$	$a_{i_1-1}^1$	$\triangledown$	$a_{i_1}^1$	$\cdots$	$a_{r_1}^1$	♦	a_1^2	a_2^2	$\cdots$	$a_{i_2-1}^2$	$\triangledown$	$a_{i_2}^2$	$\cdots$	$a_{r_2}^2$	♦	$\cdots$	♦	a_1^k	a_2^k	$\cdots$	$\triangledown$	$a_{i_k}^k$	$\cdots$	$a_{r_k}^k$	■	□	

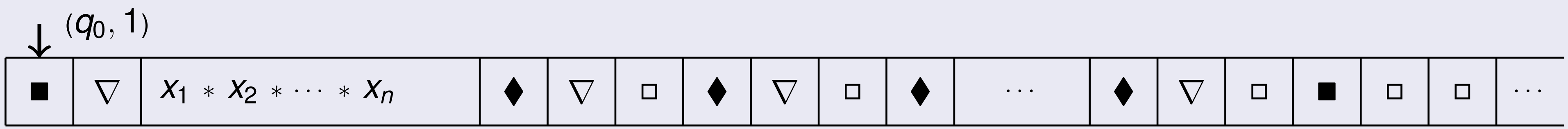
The 1-TM M' proceeds as follows.

Proof of Theorem 4.3 (cont.)

(1) At first the initial configuration



is transformed into the encoding of M's initial configuration:



Proof of Theorem 4.3 (cont.)

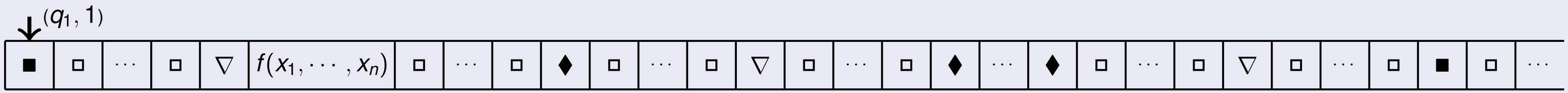
(2) Then M' simulates the TM M step by step. For simulating a single step of M, M''s head moves from left to right across the tape and stores the symbols currently read by M into M''s finite-state control.

As soon as all symbols have been read, M' moves its head back to the left, updating the symbols read and the positions of M's heads.

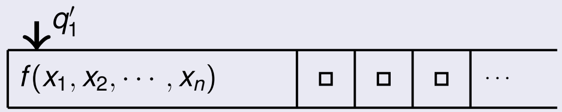
Observe that each transition of M may increase the length of the contents of some of its tapes.
Hence, M' must adjust the space provided for encoding these tapes by moving the corresponding suffix of its tape contents to the right.

Proof of Theorem 4.3 (cont.)

(3) If M does not halt on input $x_1 * x_2 * \cdots * x_n$, then neither does M'. If, however, M halts eventually, then M' halts with a tape contents of the following form:



This tape contents must now be reformated as follows:



It follows that M' computes the function f. □

A nondeterministic Turing machine (NTM) M with $k \geq 1$ tapes is given by a 7-tuple $M = (Q, \Sigma, \Gamma, \Box, \delta, q_0, q_1)$, where $Q, \Sigma, \Gamma, \Box, q_0$ and q_1 are defined as for TMs, and

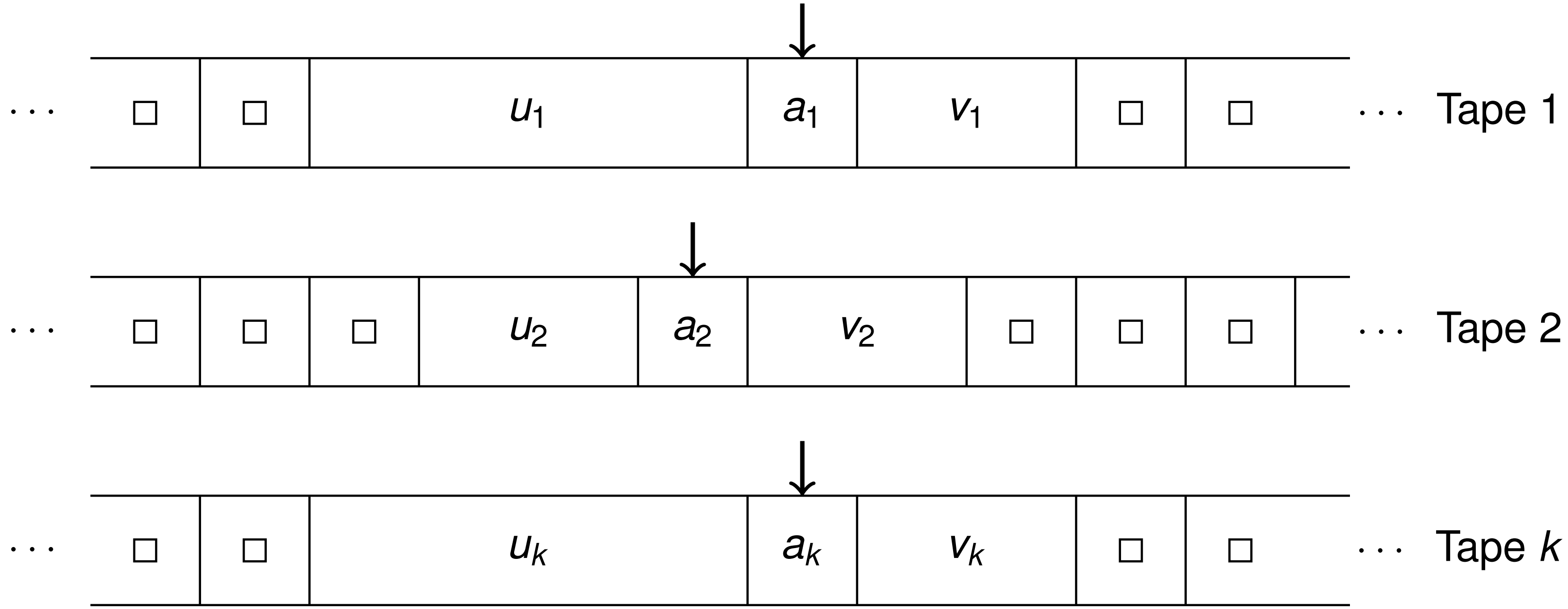
$$\delta : Q \times \Gamma^k \to 2^{Q \times \Gamma^k \times \{L,0,R\}^k}$$

is the transition relation. If

$$\delta(q, a_1, a_2, \ldots, a_k) = \{\, (q_1^{(i)}, b_1^{(i)}, \ldots, b_k^{(i)}, m_1^{(i)}, \ldots, m_k^{(i)}) \mid i = 1, \ldots, r \,\},$$

then M has r options for its next step, if it is in state q reading symbol a_j on tape j $(1 \leq j \leq k)$.

Assume that M is in state $q \in Q$ and that the tape contents and head positions are as given in the following diagram:



This configuration is encoded as

$$(u_1 q a_1 v_1, u_2 q a_2 v_2, \ldots, u_k q a_k v_k).$$

Let CONF be the set of all encodings of all configurations of M.
Then M induces a binary relation $\vdash_M$ on CONF.

The language $L(M)$ accepted by M is defined as

$$L(M) := \{ w \in \Sigma^* \mid \exists K \in \mathrm{CONF} : (q_0 w, q_0, \ldots, q_0) \vdash_M^* K, \text{ and } K \text{ contains the halting state } q_1 \}.$$

Lemma 4.4

Each k-tape-(N)TM can be simulated by a 1-(N)TM.

Theorem 4.5

For each NTM M, there exists a TM M' such that $L(M) = L(M')$.

Proof of Theorem 4.5.

W.l.o.g. let M be a 1-NTM.
We construct a 2-tape-TM M' that simulates M.

Start of the simulation: Tape 1 contains the initial configuration of M, and tape 2 is empty.

A step of the simulation: Tape 1 (or 2) contains a list of all configurations that M can reach within $2i$ $(2i+1)$ steps from the given initial configuration, and the other tape is empty.

For each configuration of M stored on tape 1 (2), all immediate successor configurations are written onto tape 2 (1).
As soon as a halting configuration is encountered, M' halts; otherwise, tape 1 (2) is erased.

Now tape 2 (1) contains all configurations that M can reach within $2i+1$ $(2i+2)$ steps from the given initial configuration, and the next step of the simulation can be executed.

It follows that $L(M') = L(M)$. □

Let $\mathcal{L}(NTM)$ denote the class of languages that are accepted by NTMs, and let $\mathcal{L}(TM)$ be the class of languages that are accepted by TMs.

Corollary 4.6

$\mathcal{L}(NTM) = \mathcal{L}(TM)$.

A language $L \subseteq \Sigma^*$ is called recursively enumerable (r.e.), if there exists a TM M such that $L = L(M)$. By RE we denote the class of all recursively enumerable languages, that is, $\mathcal{L}(NTM) = \mathcal{L}(TM) = \text{RE}$.

A language is called $L \subseteq \Sigma^*$ recursive, if there exists a TM M satisfying the following conditions:

- $\forall w \in L \quad : \quad q_0 w \vdash_M^* q_1 1.$
- $\forall w \in \Sigma^* \smallsetminus L \quad : \quad q_0 w \vdash_M^* q_1 0.$

By REC we denote the class of all recursive languages.

Theorem 4.7

(1) REC $\subseteq$ RE.

(2) *The class* REC *is closed under complementation.*

Theorem 4.8

A language L is recursive iff L and L^c are recursively enumerable.

Proof.

If L is recursive, then L and L^c are obviously r.e..

It remains to prove the converse implication.
Let M_1 be a TM such that $L(M_1) = L$, and let M_2 be a TM such that $L(M_2) = \Sigma^* \smallsetminus L$. W.l.o.g. we can assume that M_1 and M_2 are 1-TMs.

Proof of Theorem 4.8 (cont.)

Let M' be the 2-tape-TM that proceeds as follows:

(1) At first M' copies its input from tape 1 onto tape 2.

(2) On tape 1, M' simulates the TM M_1, and on tape 2, it simulates the TM M_2. These simulations are executed in parallel, step by step.

If M_1 reaches its halting state q_1, then M' produces output 1 and halts, as then $w \in L(M_1) = L$,

if M_2 reaches its halting state q_1, then M' produces the output 0 and halts, as then $w \in L(M_2) = L^c$.

Since $L(M_1) \dot{\cup} L(M_2) = \Sigma^*$, exactly one of these two cases occurs for each input $w \in \Sigma^*$. Thus, M' decides correctly whether or not w belongs to L, which shows that L is recursive. □