Automata and Grammars

SS 2018

Assignment 10: Solutions to Selected Problems

Problem 10.1. [Context-Free Grammars to PDAs]

Let $G = (\{S, A, B, C\}, \{a, b, c\}, P, S)$ be the context-free grammar (in Greibach normal form) that is given by the following set of productions:

$$\begin{array}{rcl} P & = & \{S \to aABC, S \to bBCA, A \to aAA, A \to a, \\ & & B \to bCB, B \to bB, B \to b, C \to cBC, C \to cC, C \to c\}. \end{array}$$

(1) Give a leftmost derivation for the word $w = bbccbcaaa$.

(2) Convert the grammar G_1 into a PDA M_1 using the construction from the proof of Theorem 3.19.

(3) Give the detailed computation of M_1 on input $w = bbccbcaaa$ that corresponds to the leftmost derivation from (1).

Solution. (1) A leftmost derivation for $w = bbccbcaaa$:

$$\begin{array}{llllllll} S & \to_{P_1} & bBCA & \to_{P_1} & bbCBCA & \to_{P_1} & bbcCBCA & \to_{P_1} & bbccBCA \\ & \to_{P_1} & bbccbCA & \to_{P_1} & bbccbcA & \to_{P_1} & bbccbcaAA & \to_{P_1} & bbccbcaaA \\ & \to_{P_1} & bbccbcaaa. \end{array}$$

(2) The PDA $M_1 = (\{q\}, \{a, b, c\}, \{S, A, B, C\}, \delta_1, q, S, \emptyset)$ is defined by the following transition relation:

$$\begin{array}{rcl} \delta_1(q, a, S) & = & \{(q, CBA)\}, \\ \delta_1(q, b, S) & = & \{(q, ACB)\}, \\ \delta_1(q, a, A) & = & \{(q, AA), (q, \varepsilon)\}, \\ \delta_1(q, b, B) & = & \{(q, BC), (q, B), (q, \varepsilon)\}, \\ \delta_1(q, c, C) & = & \{(q, CB), (q, C), (q, \varepsilon)\}. \end{array}$$

(3) The computation of M_1 that corresponds to the leftmost derivation in (1):

$$\begin{array}{lllll} (q, S, bbccbcaaa) & \vdash_{M_1} & (q, ACB, bccbcaaa) & \vdash_{M_1} & (q, ACBC, ccbcaaa) \\ & \vdash_{M_1} & (q, ACBC, cbcaaa) & \vdash_{M_1} & (q, ACB, bcaaa) \\ & \vdash_{M_1} & (q, AC, caaa) & \vdash_{M_1} & (q, A, aaa) \\ & \vdash_{M_1} & (q, AA, aa) & \vdash_{M_1} & (q, A, a) \\ & \vdash_{M_1} & (q, \varepsilon, \varepsilon). \end{array}$$

□

Problem 10.2. [PDA to Context-Free Grammar]

Let $M = (\{q_0, q_1\}, \{a, b\}, \{\#, Z\}, \delta, q_0, \#, \emptyset)$ be the PDA that is defined by the following transition relation:

$$\begin{array}{rclrclrcl} \delta(q_0, a, \#) &=& \{(q_1, \#Z)\}, & \delta(q_0, \varepsilon, Z) &=& \{(q_0, \varepsilon)\}, & \delta(q_0, b, \#) &=& \{(q_1, \varepsilon)\}, \\ \delta(q_1, a, Z) &=& \{(q_1, ZZ)\}, & \delta(q_1, b, Z) &=& \{(q_1, Z), (q_0, \varepsilon)\}. \end{array}$$

(1) Give an accepting computation of M for the input $w = aabbabb$.

(2) Convert the PDA M into a context-free grammar G using the construction from the proof of Theorem 3.20.

(3) Give a leftmost derivation for the word $w = aabbabb$ in G that corresponds to the computation from (1).

Solution. (1) An accepting computation of M on input $w = aabbabb$:

$$\begin{array}{rllllll} (q_0, \#, aabbabb) & \vdash_M & (q_1, \#Z, abbabb) & \vdash_M & (q_1, \#ZZ, bbabb) & \vdash_M & (q_1, \#ZZ, babb) \\ & \vdash_M & (q_0, \#Z, abb) & \vdash_M & (q_0, \#, abb) & \vdash_M & (q_1, \#Z, bb) \\ & \vdash_M & (q_0, \#, b) & \vdash_M & (q_1, \varepsilon, \varepsilon). & & \end{array}$$

(2) By the construction from the proof of Theorem 3.20 we obtain the following context-free grammar $G = (N, \{a, b\}, P, S)$ from M, where

– $N = \{S\} \cup Q \times \Gamma \times Q$
$= \{S, [q_0, \#, q_0], [q_0, \#, q_1], [q_0, Z, q_0], [q_0, Z, q_1], [q_1, \#, q_0], [q_1, \#, q_1], [q_1, Z, q_0], [q_1, Z, q_1]\}$,

– and the productions are given by the following table:

$$\begin{array}{rcl} P &=& \{S \to [q_0, \#, q_0], S \to [q_0, \#, q_1], \\ && [q_0, Z, q_0] \to \varepsilon, [q_0, \#, q_1] \to b, [q_1, Z, q_0] \to b, \\ && [q_1, Z, q_0] \to b[q_1, Z, q_0], [q_1, Z, q_1] \to b[q_1, Z, q_1], \\ && [q_0, \#, q_0] \to a[q_1, Z, q_0][q_0, \#, q_0], [q_0, \#, q_0] \to a[q_1, Z, q_1][q_1, \#, q_0], \\ && [q_0, \#, q_1] \to a[q_1, Z, q_0][q_0, \#, q_1], [q_0, \#, q_1] \to a[q_1, Z, q_1][q_1, \#, q_1], \\ && [q_1, Z, q_0] \to a[q_1, Z, q_0][q_0, Z, q_0], [q_1, Z, q_0] \to a[q_1, Z, q_1][q_1, Z, q_0], \\ && [q_1, Z, q_1] \to a[q_1, Z, q_0][q_0, Z, q_1], [q_1, Z, q_1] \to a[q_1, Z, q_1][q_1, \#, q_1]\}. \end{array}$$

(3) The leftmost derivation for the word $w = aabbabb$ that corresponds to the computation in (1):

$$\begin{array}{rllll} S & \to_P & [q_0, \#, q_1] & \to_P & a[q_1, Z, q_0][q_0, \#, q_1] \\ & \to_P & aa[q_1, Z, q_0][q_0, Z, q_0][q_0, \#, q_1] & \to_P & aab[q_1, Z, q_0][q_0, Z, q_0][q_0, \#, q_1] \\ & \to_P & aabb[q_0, Z, q_0][q_0, \#, q_1] & \to_P & aabb[q_0, \#, q_1] \\ & \to_P & aabba[q_1, Z, q_0][q_0, \#, q_1] & \to_P & aabbab[q_0, \#, q_1] \\ & \to_P & aabbabb. & & \end{array}$$

□

Problem 10.3. [Shuffle Operation]

The operation $\sqcup$ (shuffle) on Σ^* is defined as follows:

$$\forall\, u, v \in \Sigma^* : u \sqcup v := \{\, u_1 v_1 u_2 v_2 \cdots u_n v_n \mid n \geq 1, u = u_1 u_2 \cdots u_n \text{ and } v = v_1 v_2 \cdots v_n \,\}.$$

This operation is extended to sets of words (languages) as follows, where $L_1, L_2 \subseteq \Sigma^*$:

$$L_1 \sqcup L_2 = \bigcup_{u \in L_1, v \in L_2} (u \sqcup v).$$

(a) Determine the set $a^2b^2 \sqcup c^2$.

(b) Determine the set $L_1 \sqcup L_2$ for $L_1 = \{\, a^m b^m \mid m \geq 0\}$ and $L_2 = \{\, c^n \mid n \geq 0 \,\}$.

(c) Prove that $L_1 \sqcup L_2$ is a context-free language for each regular language L_1 and each context-free language L_2.

Solution. (a) $a^2b^2 \sqcup c^2 \;=\; \{ccaabb, cacabb, caacbb, caabcb, caabbc, accabb, acacbb, acabcb, acabbc, aaccbb, aacbcb, aacbbc, aabccb, aabcbc, aabbcc\}.$

(b) $L_1 \sqcup L_2 \;=\; \{\, c^{i_0} a c^{i_1} a c^{i_2} a \cdots a c^{i_n} b c^{i_{n+1}} b c^{i_{n+2}} b \cdots b c^{i_{2n}} \mid n \geq 0, i_0, i_1, \ldots, i_{2n} \geq 0 \,\}.$

(c) Let $A = (Q_1, \Sigma, \delta_1, q_0, F_1)$ be a DFA for the regular language L_1, and let

$$B = (Q_2, \Sigma, \Gamma, \delta_2, p_0, \#, F_2)$$

be a PDA for the context-free language L_2. We obtain a PDA $M = (Q, \Sigma, \Gamma, \delta, s_0, \#, F)$ for the language $L = L_1 \sqcup L_2$ by taking

- $Q = Q_1 \times Q_2$,
- $s_0 = (q_0, p_0)$,
- $F = F_1 \times F_2$,
- and by defining the transition relation δ as follows:

$$\begin{aligned}
\delta((q,p), a, Z) &= \{\, ((q,p'), \alpha) \mid (p', \alpha) \in \delta_2(p, a, Z) \,\} \cup \{((\delta_1(q,a), p), Z)\}, \\
\delta((q,p), \varepsilon, Z) &= \{\, (q,p'), \alpha) \mid (p', \alpha) \in \delta_2(p, \varepsilon, Z) \,\}
\end{aligned}$$

  for all $q \in Q_1$, $p \in Q_2$, $a \in \Sigma$, and $Z \in \Gamma$.

Let $w \in \Sigma^*$ be a given input word. Then starting from its corresponding initial configuration $((q_0, p_0), \#, w)$, M nondeterministically chooses in each step whether to simulate a step of A using the first components of its states or to simulate a step of B using the second components of its states. Finally, it accepts if it reaches a configuration of the form $((q,p), \alpha, \varepsilon)$ such that $q \in F_1$ and $p \in F_2$, which means that the simulated computations of both, A and B, are accepting. It follows that $L(M) = L(A) \sqcup L(B) = L_1 \sqcup L_2$. Thus, $L_1 \sqcup L_2$ is a context-free language. □