Theorem 3.20

The language $N(M)$ is context-free for each PDA M.

Proof.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, \#, F)$ be a PDA, and let $L = N(M)$.

W.l.o.g. we can assume the following:

For all $(q', B_1 B_2 \cdots B_k) \in \delta(q, a, A)$, we have $k \leq 2$.

Otherwise, we replace $(q', B_1 B_2 \cdots B_k) \in \delta(q, a, A)$ $(k > 2)$ by the following transitions:

$$\begin{array}{rcl} (q_1, B_1 B_2) & \in & \delta(q, a, A), \\ \{(q_2, B_2 B_3)\} & = & \delta(q_1, \varepsilon, B_2), \\ & \vdots & \\ \{(q', B_{k-1} B_k)\} & = & \delta(q_{k-2}, \varepsilon, B_{k-1}), \end{array}$$

where $q_1, q_2, \ldots, q_{k-1}$ are new states.

Proof of Theorem 3.20 (cont.)

Idea: A grammar G that simulates the computations of M through leftmost derivations.

Let $G := (N, \Sigma, P, S)$, where

$$
\begin{aligned}
N &:= \{S\} \cup [Q \times \Gamma \times Q], \\
P &:= \{\, S \to (q_0, \#, q) \mid q \in Q \,\} \\
&\cup \{\, [q, A, q'] \to a \mid (q', \varepsilon) \in \delta(q, a, A) \,\} \\
&\cup \{\, [q, A, q'] \to a[q_1, B, q'] \mid (q_1, B) \in \delta(q, a, A), q' \in Q \,\} \\
&\cup \{\, [q, A, q'] \to a[q_1, C, q_2][q_2, B, q'] \mid \\
&\qquad (q_1, BC) \in \delta(q, a, A), q', q_2 \in Q \,\}.
\end{aligned}
$$

Claim:

$\forall p, q \in Q \, \forall A \in \Gamma \, \forall x \in \Sigma^* : [q, A, p] \to^*_{\mathrm{lm}} x$ iff $(q, A, x) \vdash^*_M (p, \varepsilon, \varepsilon)$.

Proof of Theorem 3.20 (cont.)

Proof of Claim.

First we prove by induction on i that
$(q, A, x) \vdash_M^i (p, \varepsilon, \varepsilon)$ implies that $[q, A, p] \to_{\mathrm{lm}}^i x$.

For $i = 1$,
if $(q, A, x) \vdash_M (p, \varepsilon, \varepsilon)$, then $(p, \varepsilon) \in \delta(q, x, A)$ and $x \in \Sigma \cup \{\varepsilon\}$.
Hence, G contains the production $[q, A, p] \to x$, and so, $[q, A, p] \to_{\mathrm{lm}} x$.

Now assume that the above implication has been established for $i - 1$, let $x = ay$, and let

$$(q, A, ay) \vdash_M (q_1, B_2B_1, y) \vdash_M^{i-1} (p, \varepsilon, \varepsilon),$$

that is, $(q_1, B_2B_1) \in \delta(q, a, A)$.

Proof of Theorem 3.20 (cont.)

Proof of Claim (cont.)

As M can only read and replace the topmost symbol on the pushdown, it follows that $y = y_1 y_2$, and that the above computation can be factored as follows:

$$(q_1, B_1, y_1) \vdash_M^{k_1} (q_2, \varepsilon, \varepsilon) \text{ and } (q_2, B_2, y_2) \vdash_M^{k_2} (p, \varepsilon, \varepsilon),$$

where $q_2 \in Q$, and $k_1 + k_2 = i - 1$.

From the I.H. we obtain the leftmost derivations

$$[q_1, B_1, q_2] \to_{\text{lm}}^{k_1} y_1 \text{ and } [q_2, B_2, p] \to_{\text{lm}}^{k_2} y_2.$$

Proof of Theorem 3.20 (cont.)

Proof of Claim (cont.)

Because of the transition in step 1, we have

$$([q, A, p] \to a[q_1, B_1, q_2][q_2, B_2, p]) \in P,$$

and therewith we obtain the following leftmost derivation in G:

$$\begin{array}{lll} [q, A, p] & \to_{\text{lm}} & a[q_1, B_1, q_2][q_2, B_2, p] \\ & \to_{\text{lm}}^{k_1} & ay_1[q_2, B_2, p] \\ & \to_{\text{lm}}^{k_2} & ay_1y_2 = ay = x, \end{array}$$

that is, we have $[q, A, p] \to_{\text{lm}}^{i} x$.

Proof of Theorem 3.20 (cont.)

Proof of Claim (cont.)

Now we prove the converse implication, again by induction on i.

For $i = 1$, $[q, A, p] \to x$ implies that $x \in \Sigma \cup \{\varepsilon\}$ and $(p, \varepsilon) \in \delta(q, x, A)$, that is, $(q, A, x) \vdash_M (p, \varepsilon, \varepsilon)$.

Now assume that the implication has been proved for $i - 1$, and assume that

$$[q, A, p] \to a[q_1, B_1, q_2][q_2, B_2, p] \to_{\text{lm}}^{i-1} x \in \Sigma^*.$$

Then $x = ax_1x_2$, where $[q_1, B_1, q_2] \to_{\text{lm}}^{k_1} x_1$, $[q_2, B_2, p] \to_{\text{lm}}^{k_2} x_2$, and $k_1 + k_2 = i - 1$.

From the I.H. we obtain $(q_1, B_1, x_1) \vdash_M^{k_1} (q_2, \varepsilon, \varepsilon)$ and $(q_2, B_2, x_2) \vdash_M^{k_2} (p, \varepsilon, \varepsilon)$.

Proof of Theorem 3.20 (cont.)

Proof of Claim (cont.)

Hence, we have

$$(q_1, B_2B_1, x_1) \vdash_M^{k_1} (q_2, B_2, \varepsilon).$$

As $(q_1, B_2B_1) \in \delta(q, a, A)$, this yields the following computation:

$$\begin{array}{rcl} (q, A, x) & = & (q, A, ax_1x_2) \vdash_M (q_1, B_2B_1, x_1x_2) \\ & & \vdash_M^{k_1} (q_2, B_2, x_2) \vdash_M^{k_2} (p, \varepsilon, \varepsilon), \end{array}$$

that is, we have $(q, A, x) \vdash_M^{i} (p, \varepsilon, \varepsilon)$. □

Proof of Theorem 3.20 (cont.)

For all $x \in \Sigma^*$ we have the following equivalent statements:

$$\begin{array}{lll} x \in L(G) & \text{iff} & S \rightarrow^*_{\mathrm{lm}} x \\ & \text{iff} & S \rightarrow [q_0, \#, q] \rightarrow^i_{\mathrm{lm}} x \text{ for some } q \in Q \text{ and } i \geq 1 \\ & \text{iff} & (q_0, \#, x) \vdash^i_M (q, \varepsilon, \varepsilon). \end{array}$$

It follows that $L(G) = N(M)$. □

Corollary 3.21

For each language $L \subseteq \Sigma^*$, the following statements are equivalent:

(1) *$L \in \mathrm{CFL}(\Sigma)$, that is, L is generated by a context-free grammar.*

(2) *L is generated by a context-free grammar in CNF.*

(3) *L is generated by a context-free grammar in Greibach NF.*

(4) *There exists a PDA M such that $L = N(M)$.*

(5) *There exists a PDA M such that $L = L(M)$.*

3.5. Closure Properties

Theorem 3.22

The class of context-free languages CFL is closed under the operations of union, product, Kleene star, and morphism.

Proof.

(a) Let $G_1 = (N_1, \Sigma, P_1, S_1)$, $G_2 = (N_2, \Sigma, P_2, S_2)$, $N_1 \cap N_2 = \emptyset$.
For $G_3 := (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \to S_1 | S_2\}, S)$,
we have $L(G_3) = L(G_1) \cup L(G_2)$.

(b) For $G_4 := (N_1 \cup N_2 \cup \{S\}, \Sigma, P_1 \cup P_2 \cup \{S \to S_1 S_2\}, S)$,
we have $L(G_4) = L(G_1) \cdot L(G_2)$.

(c) W.l.o.g.: S_1 does not occur on the right-hand side of P_1.
For $G_5 := (N_1 \cup \{S\}, \Sigma, P_1 \cup \{S \to \varepsilon | S_1 | SS_1\} \smallsetminus \{S_1 \to \varepsilon\}, S)$,
we have $L(G_5) = (L(G_1))^*$.

Proof of Theorem 3.22 (cont.)

(d) Let $h : \Sigma^* \to \Gamma^*$ be a morphism.
For $G_6 := (N_1, \Gamma, \{ A \to h(r) \mid (A \to r) \in P_1 \}, S_1)$, where h is extended to a morphism $h : (N_1 \cup \Sigma)^* \to (N_1 \cup \Gamma)^*$ by taking $h(A) = A$ for all $A \in N_1$, we have $L(G_6) = h(L(G_1))$. □

Theorem 3.23

The class CFL is not closed under intersection nor under complement.

Proof.

$L_1 := \{ a^i b^j c^j \mid i, j > 0 \} \in$ CFL and $L_2 := \{ a^i b^i c^j \mid i, j > 0 \} \in$ CFL,
but $L_1 \cap L_2 = \{ a^i b^i c^i \mid i > 0 \} \notin$ CFL, that is,
CFL is not closed under intersection.
$L_1 \cap L_2 = \overline{(\overline{L_1} \cup \overline{L_2})}$, that is,
CFL is not closed under complement, either. □

Theorem 3.24

The class CFL is closed under intersection with regular languages, that is, if $L \in$ CFL and $R \in$ REG, then $L \cap R \in$ CFL.

Proof.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA, and
let $A = (P, \Sigma, \eta, p_0, G)$ be a DFA.
We define a PDA $M' := (Q \times P, \Sigma, \Gamma, \delta', (q_0, p_0), Z_0, F \times G)$ by taking

$$\begin{aligned} \delta'((q,p), a, A) &= \{ ((q',p'), \alpha) \mid (q',\alpha) \in \delta(q,a,A), p' = \eta(p,a) \}, \\ \delta'((q,p), \varepsilon, A) &= \{ ((q',p), \alpha) \mid (q',\alpha) \in \delta(q,\varepsilon,A) \}, \end{aligned}$$

where $q, q' \in Q$, $p, p' \in P$, $a \in \Sigma$, $A \in \Gamma$, and $\alpha \in \Gamma^*$.
Then it is easily seen that $L(M') = L(M) \cap L(A)$. □

Theorem 3.25

Let $L \subseteq \Sigma^*$ be a context-free language, and let $w \in \Sigma^*$. Then the left quotient $w \setminus L := \{\, u \in \Sigma^* \mid wu \in L \,\}$ is a context-free language.

Proof.

Let $M = (Q, \Sigma, \Gamma, \delta, q_0, Z_0, F)$ be a PDA without ε-transitions, and let $w \in \Sigma^*$. Actually, we only consider the special case that $w = b \in \Sigma$.

We define a PDA $M' := (Q \cup \{p_0\}, \Sigma, \Gamma, \delta', p_0, Z_0, F)$ by taking

$$\begin{array}{lcl} \delta'(p_0, \varepsilon, Z_0) & = & \delta(q_0, b, Z_0) \text{ and} \\ \delta'(q, a, A) & = & \delta(q, a, A) \text{ for all } q \in Q, a \in \Sigma, \text{ and } A \in \Gamma. \end{array}$$

Then $N(M') = b \setminus N(M)$. □

Remark:

We saw in Section 3 that the language

$$L := \{ a^i b^j c^k d^\ell \mid i, j, k, \ell \geq 0, \text{ and } i > 0 \text{ implies } j = k = \ell \}$$

satisfies the Pumping Lemma for context-free languages.

Claim:

L is not context-free.

Proof.

Assume that L is context-free. Then also the language

$$L' := L \cap a \cdot b^* \cdot c^* \cdot d^* = \{ ab^n c^n d^n \mid n \geq 0 \}$$

is context-free, and so is the language $a \setminus L' = \{ b^n c^n d^n \mid n \geq 0 \}$, which, however, is not context-free by the Pumping Lemma, a contradiction!

Thus, it follows that L is not context-free. □

Theorem 3.26

The class CFL is closed under inverse morphisms, that is, if $L \in \mathrm{CFL}(\Delta)$, and if $h : \Sigma^* \to \Delta^*$ is a morphism, then $h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \} \in \mathrm{CFL}(\Sigma)$.

Proof.

Let $h : \Sigma^* \to \Delta^*$ be a morphism, and
let $M = (Q, \Delta, \Gamma, \delta, q_0, Z_0, F)$ be a PDA s.t. $L(M) = L$.

We construct a PDA $M' = (Q', \Sigma, \Gamma, \delta', q_0', Z_0, F')$ for
$L' := h^{-1}(L) = \{ w \in \Sigma^* \mid h(w) \in L \}$:

- $Q' := \{ [q, x] \mid q \in Q$, and x is a suffix of $h(a), a \in \Sigma \}$,
- $q_0' := [q_0, \varepsilon]$,
- $F' := \{ [q, \varepsilon] \mid q \in F \}$, and
- δ' is defined by:

Proof of Theorem 3.26 (cont.)

$$\begin{array}{lrcl}
(1) & \delta'([q,ax],\varepsilon,A) & = & \{\,([p,x],\gamma) \mid (p,\gamma) \in \delta(q,a,A)\,\} \\
 & & & \cup\, \{\,([p,ax],\gamma) \mid (p,\gamma) \in \delta(q,\varepsilon,A)\,\}, \\
(2) & \delta'([q,\varepsilon],a,A) & = & \{([q,h(a)],A)\} \text{ for all } a \in \Sigma.
\end{array}$$

The second component of the states of M' is used as a “buffer” for storing the word $h(a)$ for a letter $a \in \Sigma$ read. This word is then processed through a simulation of a computation of M.

It follows that $h^{-1}(L) \subseteq L(M')$.

Concersely, let $w = a_1 a_2 \cdots a_n \in L(M')$.

A transition reading a letter $a \in \Sigma$ can only be applied if and when the buffer is empty, that is, each accepting computation of M' on input w can be written as follows:

Proof of Theorem 3.26 (cont.)

$([q_0, \varepsilon], Z_0, a_1 a_2 \cdots a_n)$

$\vdash^*_{M'}$ $([p_1, \varepsilon], \alpha_1, a_1 a_2 \cdots a_n)$ (simulating ε-transitions of M)

$\vdash_{M'}$ $([p_1, h(a_1)], \alpha_1, a_2 \cdots a_n)$ ($h(a_1)$ is “read”)

$\vdash^*_{M'}$ $([p_2, \varepsilon], \alpha_2, a_2 \ldots a_n)$ (for $(p_1, \alpha_1, h(a_1)) \vdash^*_M (p_2, \alpha_2, \varepsilon)$)

$\vdots$

$\vdash_{M'}$ $([p_n, h(a_n)], \alpha_n, \varepsilon)$ ($h(a_n)$ is “read”)

$\vdash^*_{M'}$ $([p_{n+1}, \varepsilon], \alpha_{n+1}, \varepsilon)$ (for $(p_n, \alpha_n, h(a_n)) \vdash^*_M (p_{n+1}, \alpha_{n+1}, \varepsilon)$).

It follows that

$(q_0, Z_0, h(a_1 a_2 \cdots a_n)) = (q_0, Z_0, h(a_1)h(a_2) \cdots h(a_n)) \vdash^*_M (p_{n+1}, \alpha_{n+1}, \varepsilon)$.

As M' accepts, $p_{n+1} \in F$, which implies that M accepts, too.

Thus, $L(M') = h^{-1}(L)$, which yields $h^{-1}(L) \in \text{CFL}(\Sigma)$. □

In analogy to Corollary 2.29 we have the following closure property.

Corollary 3.27

The class CFL *is closed under finite transductions, that is, if* $T \subseteq \Sigma^* \times \Delta^*$ *is a finite transduction, and if* $L \in \mathrm{CFL}(\Sigma)$, *then* $T(L) \in \mathrm{CFL}(\Delta)$.