Automata and Grammars

SS 2018

Assignment 7: Solutions to Selected Problems

Problem 7.1 [Pumping Lemma]

Prove that the following languages are not regular by applying the Pumping Lemma for regular languages (Theorem 2.34):

(a) $L_a = \{ a^m b^n \mid n > m \geq 1 \}$,
(b) $L_b = \{ a^m b^n \mid m \leq n \leq 2m \}$,
(c) $L_c = \{ w \in \{a,b\}^* \mid |w|_a = |w|_b \}$,
(d) $L_d = \{ a^{2^n} \mid n \geq 0 \}$.

Solution. (a) Assume that L_a is regular. Then it satisfies the Pumping Lemma, that is, there exists a constant $c \geq 1$ such that each word $x \in L_a$, $|x| \geq c$, admits a factorization $x = uvw$ satisfying $|uv| \leq c$, $v \neq \varepsilon$, and $uv^iw \in L_a$ for all $i \geq 0$.
Consider the word $x = a^c b^{c+1}$. Then $x \in L_a$ and $|x| = 2c+1 > c$. Hence, x has a factorization $x = uvw$ that satisfies the three properties above. Now $a^c b^{c+1} = x = uvw$, where $|uv| \leq c$, implies that $u = a^r$ and $v = a^s$ for some r, s satisfying $s > 0$ and $r + s \leq c$. Hence, $w = a^{c-r-s} b^{c+1}$. Consider the word $uv^2w = a^r a^s a^s a^{c-r-s} b^{c+1} = a^{c+s} b^{c+1}$. As $s > 0$, we see that $uv^2w \notin L_a$, a contradiction! Thus, L_a does not satisfy the Pumping Lemma, which proves that L_a is NOT regular.

(d) Assume that L_d is regular, and let $c \geq 1$ be the corresponding constant from the Pumping Lemma. Consider the word $x = a^{2^c}$. Then $x \in L_d$ and $|x| = 2^c > c$. Hence, x has a factorization $x = uvw$ that satisfies the three properties stated in the Pumping Lemma. Now $a^{2^c} = x = uvw$, where $|uv| \leq c$, implies that $u = a^r$ and $v = a^s$ for some r, s satisfying $s > 0$ and $r + s \leq c$. Hence, $w = a^{2^c - r - s}$. Consider the word $uv^2w = a^r a^s a^s a^{2^c - r - s} = a^{2^c + s}$. As $0 < s \leq c$, we see that $2^c < 2^c + s < 2^c + 2^c = 2^{c+1}$, that is, $uv^2w \notin L_d$, a contradiction! Thus, L_d does not satisfy the Pumping Lemma, which proves that L_d is NOT regular. □

Problem 7.2. [Pumping Lemma]

Let L be the following language over $\Sigma = \{a, b\}$:

$$L = \{ ab(ba)^m ba^n ba^n \mid m, n \geq 1 \} \cup \{ abba^m ba^n \mid m, n \geq 1 \}.$$

(a) Prove that L satisfies the Pumping Lemma for regular languages (Theorem 2.34) by determining a corresponding constant $c \geq 1$.

(b) Prove that the language L is not regular by using the Theorem of Myhill-Nerode (Theorem 2.12).

(c) Is the language $L_c = \{ (ab)^n a (ba)^n \mid n \geq 1 \}$ regular? Provide a proof for your answer!

Solution. (a) Choose $c = 7$, and let $x \in L$ such that $|x| \geq 7$. If $x = ab(ba)^m ba^n ba^n$ for some $m, n \geq 1$, we choose $u = ab$, $v = ba$, and $w = (ba)^{m-1} ba^n ba^n$. Then $|uv| = |abba| = 4 < c$, $v \neq \varepsilon$, and $uv^iw = ab(ba)^i (ba)^{m-1} ba^n ba^n = ab(ba)^{m-1+i} ba^n ba^n$. If $m \geq 2$, then $m - 1 + i \geq m - 1 \geq 1$, and hence, $uv^iw \in L$. If $m = 1$, then $m - 1 + i = i$, and so $uv^iw \in L$ for all $i \geq 1$. Finally, for $i = 0$, we have $uv^iw = ab(ba)^0 ba^n ba^n = abba^n ba^n \in L$.

If $x = abba^m ba^n$ for some $m, n \geq 1$, then $m \geq 2$ or $n \geq 2$. In the former case we choose $u = abb$, $v = a$, and $w = a^{m-1}ba^n$. Then $|uv| = |abba| = 4 < c$, $v \neq \varepsilon$, and $uv^i w = abba^i a^{m-1} ba^n = abba^{m-1+i}ba^n \in L$, as $m - 1 + i \geq m - 1 \geq 1$. If $m = 1$, then $n \geq 2$, and we choose $u = abbab$, $v = a$, and $w = a^{n-1}$. Then $|uv| = |abbaba| = 6 < c$, $v \neq \varepsilon$, and $uv^i w = abbaba^i a^{n-1} = abbaba^{n-1+i} \in L$, as $n - 1 + i \geq n - 1 \geq 1$. Hence, we see that L satisfies the Pumping Lemma for regular languages with the constant $c = 7$.

Remark. The constant $c = 6$ is not sufficient, as the word $x = abbaba$ of length 6 does not admit a corresponding factorization.

(b) For each $n \geq 2$, we consider the word $x_n = ab(ba)^2 ba^n b$. Then $x_n z \in L$ iff $z = a^n$. This means that $(x_n, x_k) \notin R_L$ for all $n, k \geq 2$, $n \neq k$. Thus, the Nerode relation R_L has infinite index, which shows that L is not regular.

(c) As $(ab)^n a (ba)^n = (ab)^{2n} a$, we see that $L_c = \{ (ab)^{2n} a \mid n \geq 1 \}$ is described by the regular expression $r = (abab)^+ a = (abba)(abba)^* a$. Thus, L_c is a regular language. □

Problem 7.3. [Decision Problems]

Determine the cardinality of the language $L(A_i)$ for the following NFAs A_i $(1 \leq i \leq 4)$:

(a) $A_1 = (Q, \{a, b\}, \delta_1, \{0, 5\}, \{6\})$, where δ_1 is defined as follows:

δ_1	$0^{\leftarrow}$	1	2	3	4	$5^{\leftarrow}$	$6_{\rightarrow}$	7	8
a	4	2	3	1	0	7	3	1	3
b	1	3	2	3	4	8	–	8	6

(b) $A_2 = (Q, \{a, b\}, \delta_2, \{0\}, \{5, 10\})$, where δ_2 is defined as follows:

δ_2	$0^{\leftarrow}$	1	2	3	4	$5_{\rightarrow}$	6	7	8	9	$10_{\rightarrow}$
a	1	3	1	4	2	4	2	6	8	6	10
b	7	1	–	3	4	10	0	9	7	–	5

(c) $A_3 = (Q, \{a, b\}, \delta_3, \{0\}, \{0, 5, 7\})$, where δ_3 is defined as follows:

δ_3	$0^{\leftarrow}_{\rightarrow}$	1	2	3	4	$5_{\rightarrow}$	6	$7_{\rightarrow}$	8	9
a	1, 2	–	–	2	6	4	2, 7	–	5, 9	9
b	4	1, 3	2	–	–	3	–	8, 9	–	9

(d) $A_4 = (Q, \{a, b\}, \delta_4, \{0, 5\}, \{10\})$, where δ_4 is defined as follows:

δ_4	$0^{\leftarrow}$	1	2	3	4	$5^{\leftarrow}$	6	7	8	9	$10_{\rightarrow}$
a	1	2	3	4	3	–	–	–	3, 9	–	–
b	–	–	–	9	–	6	7	8	–	1, 6, 10	–

Solution. (a) The final state 6 is reachable from the initial state 5. Hence, $L(A_1) \neq \emptyset$. In fact, $L(A_1) = \{abb, bb\}$, that is, it has cardinality 2.

(b) The states $5, 8, 10$ are not accessible in A_2. Hence, $L(A_2) = \emptyset$, as 5 and 10 are the only final states.

(c) As 0 is the initial and a final state, we see that $L(A_3) \neq \emptyset$. In fact, $L(A_3)$ contains all words of the form $baa(baaaa)^m$, $m \geq 0$, as

$$0 \to_b 4 \to_a 6 \to_a 7 \to_b 8 \to_a 5 \to_a 4.$$

Thus, $L(A_3)$ is infinite.

(d) $L(A_4)$ contains all words of the form $a(aabb)^m aabb$ $(m \geq 0)$, and hence, it is infinite. □