Automata and Grammars

SS 2018

Assignment 6: Solutions to Selected Exercises

Problem 6.1 [Right and Left Quotients]

Let L and P be two languages over Σ. The *right quotient* L/P of L by P is defined as

$$L/P = \{ u \in \Sigma^* \mid \exists v \in P : uv \in L \}$$

and the *left quotient* $P \backslash L$ of L by P is defined as

$$P \backslash L = \{ v \in \Sigma^* \mid \exists u \in P : uv \in L \}.$$

Let $A = (Q, \Sigma, \delta, I, F)$ be an NFA such that $L(A) = L$.

(a) Prove that the right quotient L/P is a regular language by showing how to obtain an NFA for L/P from A and P.

(b) Prove that the left quotient $P \backslash L$ is a regular language by showing how to obtain an NFA for $P \backslash L$ from A and P.

Solution.

(a) Let $A = (Q, \Sigma, \delta, I, F)$ be an NFA such that $L(A) = L$. Let

$$Q_P = \{ q \in Q \mid \exists v \in P : \hat{\delta}(q, v) \cap F \neq \emptyset \},$$

that is, Q_P contains those states of A from which A can reach a final state by reading an element of P. Then $A_P = (Q, \Sigma, \delta, I, Q_P)$ is an NFA such that $L(A_P) = L/P$.

(b) Let

$$Q'_P = \{ q \in Q \mid \exists u \in P : q \in \hat{\delta}(I, u) \},$$

that is, Q'_P contains those states of A which A can reach from an initial state by reading a word from P. Then $A'_P = (Q, \Sigma, \delta, Q'_P, F)$ is an NFA such that $L(A'_P) = P \backslash L$. □

Problem 6.2. [Regular Expressions]

(a) Write a regular expression that defines the language on $\{a, b\}$ that contains exactly those words that begin with 'ba' and that end with 'ab'.

(b) Write a regular expression that defines the language on $\{a, *\}$ that contains exactly all those words of the form a, $a * a$, $a * a * a$,...

(c) Convert the following regular expressions into equivalent ε-NFAs by using the construction from the proof of Theorem 2.30:

- $(ab + c)^*$,
- $((ab + c)^* a (bc)^* + b)^*$.

Solution. (a) $r_a = ((ba(a+b)^*ab) + bab)$.
(b) $r_b = (a(*a)^*)$. $\square$

Problem 6.3. [Regular Expressions]

Convert the NFA $A = (\{q_0, q_1, q_2\}, \{a, b\}, q_0, \{q_1, q_2\}, \delta)$ into an equivalent regular expression using the construction from the proof of Theorem 2.31, where δ is defined as follows:

δ	$q_0^{\leftarrow}$	$q_{1\rightarrow}$	$q_{2\rightarrow}$
a	q_0	q_2	q_0
b	q_1	q_1	q_1

Solution.

$$
\begin{array}{lllllllll}
r^0_{0,0} & = & a+\varepsilon, & r^0_{0,1} & = & b, & r^0_{0,2} & = & \emptyset, \\
r^0_{1,0} & = & \emptyset, & r^0_{1,1} & = & b+\varepsilon, & r^0_{1,2} & = & a, \\
r^0_{2,0} & = & a, & r^0_{2,1} & = & b, & r^0_{2,2} & = & \varepsilon, \\
r^1_{0,0} & = & a^*, & r^1_{0,1} & = & a^*b, & r^1_{0,2} & = & \emptyset, \\
r^1_{1,0} & = & \emptyset, & r^1_{1,1} & = & b+\varepsilon, & r^1_{1,2} & = & a, \\
r^1_{2,0} & = & a^+, & r^1_{2,1} & = & a^*b, & r^1_{2,2} & = & \varepsilon, \\
r^2_{0,0} & = & a^*, & r^2_{0,1} & = & a^*b^+, & r^2_{0,2} & = & a^*b^+a, \\
r^2_{1,0} & = & \emptyset, & r^2_{1,1} & = & b^*, & r^2_{1,2} & = & b^*a, \\
r^2_{2,0} & = & a^+, & r^2_{2,1} & = & a^*b^+, & r^2_{2,2} & = & a^*b^+a+\varepsilon, \\
r^3_{0,1} & = & (a^*b^+a)^*a^*b^+, \\
r^3_{0,2} & = & (a^*b^+a)^+, \\
r_a & = & (a^*b^+a)^*a^*b^+ + (a^*b^+a)^+.
\end{array}
$$

$\square$

Problem 6.4. [Regular Expressions]

Prove that the following identities hold for all regular expressions r, s, t:

$$
\begin{array}{llll}
(1) & r+\emptyset & = & r, \\
(2) & r\varepsilon & = & r, \\
(3) & r\emptyset & = & \emptyset, \\
(4) & r(s+t) & = & (rs)+(rt), \\
(5) & \emptyset^* & = & \varepsilon, \\
(6) & (r^*)^* & = & r^*, \\
(7) & (r+s)^* & = & (r^*+s^*)^*,
\end{array}
$$

Hint: For proving (1) one must show that $L(r+\emptyset) = L(r)$ holds for each regular expression r, and analogously for the other identities.

Solution. Let r, s, t be regular expressions over some alphabet Σ.

(1) $L(r+\emptyset) = L(r) \cup L(\emptyset) = L(r) \cup \emptyset = L(r)$.

(2) $L(r\varepsilon) = L(r) \cdot L(\varepsilon) = L(r) \cdot \{\varepsilon\} = L(r)$.

(3) $L(r\emptyset) = L(r) \cdot L(\emptyset) = L(r) \cdot \emptyset = \emptyset$.

(4)
$$
\begin{array}{lll}
L(r(s+t)) & = & L(r) \cdot L(s+t) \\
& = & L(r) \cdot (L(s) \cup L(t)) \\
& = & \{\, uv \mid u \in L(r), v \in L(s) \cup L(t) \,\} \\
& = & \{\, uv \mid u \in L(r), v \in L(s) \,\} \cup \{\, uv \mid u \in L(r), v \in L(t) \,\} \\
& = & (L(r) \cdot L(s)) \cup (L(r) \cdot L(t)) \\
& = & L(rs) \cup L(rt) \\
& = & L((rs)+(rt)).
\end{array}
$$

(5) $L(\emptyset^*) = (L(\emptyset))^* = \{\emptyset\}^* = \{\varepsilon\} = L(\varepsilon)$.

(6) Obviously, $L(r) \subseteq L(r^*)$, and hence, $L(r^*) = (L(r))^* \subseteq (L(r^*))^* = L((r^*)^*)$. Conversely,

$$\begin{array}{rcl} L((r^*)^*) & = & (L(r^*))^* \\ & = & ((L(r))^*)^* \\ & = & \bigcup_{n\geq 0}((L(r))^*)^n \\ & = & \bigcup_{n\geq 0}\{\, w_1 w_2 \cdots w_n \mid w_1, w_2, \ldots, w_n \in (L(r))^* \,\} \\ & \subseteq & \bigcup_{n\geq 0}\{\, w_1 w_2 \cdots w_n \mid w_1, w_2, \ldots, w_n \in L(r) \,\} \\ & = & (L(r))^* = L((r^*), \end{array}$$

which shows that $L(r^*) = L((r^*)^*)$.

(7) Obviously,

$$L(r+s) = L(r) \cup L(s) \subseteq (L(r))^* \cup (L(s))^* = L(r^* + s^*),$$

and hence,

$$L((r+s)^*) = (L(r+s))^* \subseteq (L(r^* + s^*))^* = L((r^* + s^*)^*).$$

On the other hand,

$$L(r^* + s^*) = L(r^*) \cup L(s^*) = (L(r))^* \cup (L(s))^* \subseteq (L(r) \cup L(s))^* = L((r+s)^*),$$

and hence,

$$L((r^* + s^*)^*) = (L(r^* + s^*))^* \subseteq (L((r+s)^*))^* = L(((r+s)^*)^*) = L((r+s)^*)$$

by (6). Thus, $L((r+s)^*) = L((r^* + s^*)^*)$. □