Automata and Grammars

SS 2018

Assignment 5: Solutions to Selected Problems

Problem 5.1 [2DFA]

Let $A = (Q, \Sigma, \rhd, \lhd, \delta, q_0, F)$ be the 2DFA with initial state p_0 and final state q_4 that is given by the following table:

$\delta:$	p_0	$\overleftarrow{q_0}$	$\overleftarrow{q_1}$	$\overleftarrow{q_2}$	$\overleftarrow{q_3}$	$\overrightarrow{q_0}$	$\overrightarrow{q_1}$	$\overrightarrow{q_2}$	$\overrightarrow{q_3}$	q_4
a	(p_0, R)	$(\overleftarrow{q_1}, L)$	$(\overleftarrow{q_2}, L)$	$(\overleftarrow{q_3}, L)$	$(\overleftarrow{q_2}, L)$	$(\overrightarrow{q_1}, R)$	$(\overrightarrow{q_2}, R)$	$(\overrightarrow{q_3}, R)$	$(\overrightarrow{q_2}, R)$	(q_4, R)
b	(p_0, R)	$(\overleftarrow{q_0}, L)$	$(\overleftarrow{q_0}, L)$	$(\overleftarrow{q_2}, L)$	$(\overleftarrow{q_3}, L)$	$(\overrightarrow{q_0}, R)$	$(\overrightarrow{q_0}, R)$	$(\overrightarrow{q_2}, R)$	$(\overrightarrow{q_3}, R)$	(q_4, R)
c	$(\overleftarrow{q_0}, L)$	$-$	$-$	$-$	$-$	$-$	$-$	(q_4, R)	(q_4, R)	$-$
$\rhd$	(p_0, R)	$(\overrightarrow{q_0}, R)$	$(\overrightarrow{q_1}, R)$	$(\overrightarrow{q_2}, R)$	$(\overrightarrow{q_3}, R)$	$-$	$-$	$-$	$-$	$-$
$\lhd$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$

(a) Describe the step-by-step computation of A on input $abacba$.

(b) Determine all seven crossing sequences of the above computation.

(c) Which language is accepted by A?

Solution.

(a) On input $w = abacba$, the 2DFA A executes the following computation:

$$\begin{array}{lllllll}
p_0 \rhd abacba \lhd & \vdash_A & \rhd p_0 abacba \lhd & \vdash_A & \rhd a p_0 bacba \lhd & \vdash_A & \rhd ab p_0 acba \lhd \\
 & \vdash_A & \rhd aba p_0 cba \lhd & \vdash_A & \rhd ab \overleftarrow{q_0} acba \lhd & \vdash_A & \rhd a \overleftarrow{q_1} bacba \lhd \\
 & \vdash_A & \rhd \overleftarrow{q_0} abacba \lhd & \vdash_A & \overleftarrow{q_1} \rhd abacba \lhd & \vdash_A & \rhd \overrightarrow{q_1} abacba \lhd \\
 & \vdash_A & \rhd a \overrightarrow{q_2} bacba \lhd & \vdash_A & \rhd ab \overrightarrow{q_2} acba \lhd & \vdash_A & \rhd aba \overrightarrow{q_3} cba \lhd \\
 & \vdash_A & \rhd abac q_4 ba \lhd & \vdash_A & \rhd abacb q_4 a \lhd & \vdash_A & \rhd abacba q_4 \lhd,
\end{array}$$

and as q_4 is the final state, we see that A accepts input $w = abacba$.

(b) The above computation yields the following crossing sequences:

$\rhd$		a		b		a		c		b		a		$\lhd$
	p_0		p_0		p_0		p_0		q_4		q_4		q_4	
	$\overleftarrow{q_1}$		$\overleftarrow{q_0}$		$\overleftarrow{q_1}$		$\overleftarrow{q_0}$							
	$\overrightarrow{q_1}$		$\overrightarrow{q_2}$		$\overrightarrow{q_2}$		$\overrightarrow{q_3}$							

(c) The language $L(A)$ is the set of all words of the form wcu, where $w, u \in \{a, b\}^*$ such that w begins with a or it contains aa as a factor. $\square$

Problem 5.2 [Moore Automaton]

Design a Moore automaton with input and output alphabet $\{0, 1\}$ that realizes the function f that is defined as follows:

$$f(a_m a_{m-1} \cdots a_2 a_1) = \begin{cases} 1, & \text{if } \sum_{i=1}^{m} a_i \cdot 2^{i-1} \equiv 0 \mod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Solution. Let $A = (\{q_0, q_1, q_2\}, \{0,1\}, \{0,1\}, \delta, \sigma, q_0)$ be the Moore automaton that is given by the following table:

δ	q_0	q_1	q_2
0	q_1	q_2	q_2
1	q_0	q_0	q_0
σ	0	0	1

Then
$\delta(q_0, u) = q_0$ iff $u = \varepsilon$ or u ends in 1,
$\delta(q_0, u) = q_1$ iff $u = 0$ or u ends in 1^+0,
$\delta(q_0, u) = q_2$ iff u ends in 00.
As $f(u) = 1$ iff u ends in 00, we see that A computes the intended function. □

Problem 5.3 [Mealey Automaton]

Design a Mealey automaton with input and output alphabet $\{0,1\}$ that realizes the following function:

- output 1, if the current input symbol is a part of a sequence of 1s, which is directly preceded by the factor 00,
- output 0, otherwise.

Solution. Let $A = (\{q_0, q_1, q_2, q_3\}, \{0,1\}, \{0,1\}, \delta, \sigma, q_0)$ be the Mealey automaton that is given by the following table:

(δ, σ)	q_0	q_1	q_2	q_3
0	$(q_1, 0)$	$(q_2, 0)$	$(q_2, 0)$	$(q_1, 0)$
1	$(q_0, 0)$	$(q_0, 0)$	$(q_3, 1)$	$(q_3, 1)$

$\delta(q_0, u) = q_0$ iff $u \in 1^*$ or $u \in 01^+$ or u ends in 101^+,
$\delta(q_0, u) = q_1$ iff $u = 0$ or u ends in 1^+0,
$\delta(q_0, u) = q_2$ iff u ends in 00, and
$\delta(q_0, u) = q_3$ iff u ends in 001^+.
It is now easily seen that A computes the intended function. □

Problem 5.4 [Moore and Mealey Automata]

Let $A = (Q, \Sigma, \Delta, \delta, \sigma, q_0)$ be a Moore automaton and let $B = (Q', \Sigma, \Delta, \delta', \sigma', q_0')$ be a Mealey automaton. For $u \in \Sigma^*$, let $F_A(u) \in \Delta^*$ and $F_B(u) \in \Delta^*$ be the output words that are generated by A and B on input u. The automata A und B are called *equivalent*, if $F_A(u) = \sigma(q_0) \cdot F_B(u)$ for all $u \in \Sigma^*$. Prove the following statements:

(a) For each Moore automaton, there exists an equivalent Mealey automaton.

(b) For each Mealey automaton, there exists an equivalent Moore Automaton.

Solution. (a) Let $A = (Q, \Sigma, \Delta, \delta, \sigma, q_0)$ be a Moore automaton. We define a Mealey automaton $B = (Q, \Sigma, \Delta, \delta, \sigma', q_0)$ by taking $\sigma'(q, a) = \sigma(\delta(q, a))$ for all $q \in Q$ and all $a \in \Sigma$. If A is in state q and reads letter a, then it changes to state $\delta(q, a)$, which then yields the output $\sigma(\delta(q, a))$. Now if B is in state q and reads the letter a, then it changes to state $\delta(q, a)$ producing the output $\sigma'(q, a) = \sigma(\delta(q, a))$ during this transition. Hence, it follows that B is equivalent to A.

(b) Let $B = (Q, \Sigma, \Delta, \delta, \sigma, q_0)$ be a Mealey automaton. We define a Moore automaton $A = (Q', \Sigma, \Delta, \delta', \sigma', q_0')$ as follows:

- $Q' = Q \times \Delta = \{\, (q,b) \mid q \in Q, b \in \Delta \,\}$,
- $q_0' = (q_0, b_0)$ for some letter $b_0 \in \Delta$,
- $\sigma'((q,b)) = b$ for all $(q,b) \in Q \times \Delta$, and
- δ' is defined through $\delta'((q,b),a) = (\delta(q,a), \sigma(q,a))$ for all $q \in Q$, $b \in \Delta$, and $a \in \Sigma$.

If B is in state q and reads the letter a, then it changes to state $\delta(q,a)$ producing the output $\sigma(q,a)$. Now if A is in a state of the form (q,b) and reads the letter a, then it changes to the state $(\delta(q,a), \sigma(q,a))$, which then yields the output $\sigma'((\delta(q,a), \sigma(q,a)) = \sigma(q,a)$. Hence, it follows that A is equivalent to B. □

Problem 5.5 [Finite-State Transducer]

A regular substitution $\varphi : \Sigma^* \to 2^{\Gamma^*}$ maps each letter $a \in \Sigma$ to a regular language $\varphi(a) \in \mathsf{REG}(\Gamma)$ and then $\varphi(a_1 a_2 \cdots a_n) = \varphi(a_1) \cdot \varphi(a_2) \cdots \varphi(a_n)$.

(a) Let $\varphi_1 : \{a,b\}^* \to 2^{\{a,b\}^*}$ be the regular substitution that is given through $a \mapsto \{a\}^*$ and $b \mapsto \{b\}^*$. Construct a finite-state transducer T_1 such that $T_1(w) = \varphi_1(w)$ for each word $w \in \{a,b\}^*$.

(b) Prove that each regular substitution can be realized by a finite-state transducer, that is, if $\varphi : \Sigma^* \to 2^{\Gamma^*}$ is a regular substitution, then there exists a finite-state transducer T such that $T(w) = \varphi(w)$ for each word $w \in \Sigma^*$.

Solution. (a) Let $T_1 = (\{q_0, q_a, q_b\}, \{a,b\}, \{a,b\}, \delta, q_0, \{q_0\})$ be the finite-state transducer that is given by the following table:

δ	q_0	q_a	q_b
ε	$-$	$\{(q_a,a),(q_0,\varepsilon)\}$	$\{(q_b,b),(q_0,\varepsilon)\}$
a	$\{(q_a,\varepsilon)\}$	$-$	$-$
b	$\{(q_b,\varepsilon)\}$	$-$	$-$

On input a word $w \in \{a,b\}^*$, T_1 reads w letter by letter, and for each letter read, it can output any number of copies of that letter. It follows that $T_1(w) = \varphi(w)$.

(b) Let $\varphi : \Sigma^* \to 2^{\Gamma^*}$ be a regular substitution. Then $\varphi(a)$ is a regular language over Γ for each letter $a \in \Sigma$. So there exists a DFA $A_a = (Q_a, \Gamma, \delta_a, q_0^{(a)}, F_a)$ such that $L(A_a) = \varphi(a)$ for each $a \in \Sigma$. Without loss of generality we can assume that $Q_a \cap Q_b = \emptyset$ for all $a \neq b$. Now we define a finite-state transducer $T = (Q, \Sigma, \Gamma, \delta, q_0, F)$ as follows:

- $Q = \{q_0\} \cup \bigcup_{a \in \Sigma} Q_a$,
- $F = \{q_0\}$,
- and δ is defined as follows:

$$\begin{array}{lcll} \delta(q_0, a) & = & \{(q_0^{(a)}, \varepsilon)\} & \text{for each } a \in \Sigma, \\ \delta(q^{(a)}, \varepsilon) & = & \{(\delta_a(q^{(a)}, b), b) \mid b \in \Gamma\,\} & \text{for all } a \in \Sigma \text{ and } q^{(a)} \in Q_a \smallsetminus F_a, \\ \delta(q^{(a)}, \varepsilon) & = & \{(\delta_a(q^{(a)}, b), b) \mid b \in \Gamma\,\} \cup \{(q_0, \varepsilon)\} & \text{for all } a \in \Sigma \text{ and } q^{(a)} \in F_a. \end{array}$$

Let $w = a_1 a_2 \cdots a_n \in \Sigma^n$. On input w, T proceeds as follows:

$$\begin{array}{rcl} (q_0 w, \varepsilon) & = & (q_0 a_1 a_2 \cdots a_n, \varepsilon) \quad \vdash_T \quad (q_0^{(a_1)} a_2 \cdots a_n, \varepsilon) \\ & \vdash_T^* & (q_f^{(a_1)} a_2 \cdots a_n, u_1) \quad \vdash_T \quad (q_0 a_2 \cdots a_n, u_1) \\ & \vdash_T^* & (q_f^{(a_n)}, u_1 u_2 \cdots u_n) \quad \vdash_T \quad (q_0, u_1 u_2 \cdots u_n), \end{array}$$

where $u_i \in \varphi(a_i)$ and $q_f^{(a_i)} \in F_{a_i}$. It follows that $T(w) = \varphi(w)$. □