Automata and Grammars

SS 2018

Assignment 3: Solutions to Selected Problems

Seminary: Thursday, March 15, 2018.

Problem 3.1 [Nerode Relation]

Determine all equivalence classes of the Nerode relations $R_L \subseteq \Sigma^* \times \Sigma^*$ for the following languages:

(a) $\Sigma = \{a, b\}$ and $L = \{a, aab, abb\}$,
(b) $\Sigma = \{a, b\}$ and $L = \{ a^m b a^n \mid m, n \geq 1 \}$,
(c) $\Sigma = \{a, b\}$ and $L = \{ a^n b a^n \mid n \geq 0 \}$,

In addition, for those language L, for which R_L has finite index, construct the equivalence class automaton for L.

Solution. We denote the equivalence class of a word w simply by $[w]$.

(b) For $L = \{ a^m b a^n \mid m, n \geq 1 \}$, we obtain the following equivalence classes:

$$[\varepsilon] = \{\varepsilon\}, \quad [a] = \{ a^m \mid m \geq 1 \}, \quad [ab] = \{ a^m b \mid m \geq 1 \},$$
$$[aba] = L, \quad [b] = \{ w \in \{a, b\}^+ \mid w \text{ is not a prefix of any element of } L \} .$$

As R_L has index 5, we obtain the following equivalence class automaton $(Q, \{a, b\}, \delta, [\varepsilon], \{[aba]\})$ for L, where δ is described by the following table:

	$[\varepsilon]$	$[a]$	$[ab]$	$[aba]$	$[b]$
a	$[a]$	$[a]$	$[aba]$	$[aba]$	$[b]$
b	$[b]$	$[ab]$	$[b]$	$[b]$	$[b]$

(c) For $L = \{ a^n b a^n \mid n \geq 0 \}$, we obtain the following equivalence classes:

$$[\varepsilon] = \{\varepsilon\},$$
$$[a^i] = \{a^i\} \quad \text{for all } i \geq 1,$$
$$[a^i b] = \{a^i b\} \quad \text{for all } i \geq 0,$$
$$[a^i b a^j] = \{ a^{i+k} b a^{j+k} \mid k \geq 0 \} \quad \text{for all } 0 < j < i,$$
$$[b] = L,$$
$$[bb] = \{ w \in \{a, b\}^+ \mid w \text{ is not a prefix of any element of } L \}.$$

Thus, R_L has infinite index. □

Problem 3.2 [Myhill Nerode Theorem]

Use the Myhill Nerode Theorem to check which of the following languages are accepted by DFAs:

(a) $L_a = \{ a^n b^n \mid n \geq 1 \}$,
(b) $L_b = \{ ww \mid w \in \{a, b\}^* \}$,
(c) $L_c = \{ a^{2n} \mid n \geq 0 \}$,
(d) $L_d = \{ a^{2^n} \mid n \geq 0 \}$.

Solution.

(a) For each $n \geq 1$, let $w_n = a^n b$. Then $w_n z \in L_a$ iff $z = b^{n-1}$. Hence, $(w_m, w_n) \notin R_{L_a}$ for all $m \neq n$, that is, R_{L_a} has infinite index. Thus, by the Myhill Nerode Theorem L_a is not accepted by any DFA.

(c) It is easily seen that $[\varepsilon] = \{ a^{2n} \mid n \geq 0 \} = L_c$ and $[a] = \{ a^{2n+1} \mid n \geq 0 \}$, which shows that R_{L_c} has index two. Thus, L_c is accepted by a DFA.

(d) For each $n \geq 0$, let $w_n = a^{2^n}$. Then $w_n w_n = a^{2^n + 2^n} = a^{2^{n+1}} \in L_d$, but $w_{n+k} w_n = a^{2^{n+k}+2^n} = a^{2^n \cdot (2^k+1)} \notin L_d$. Thus, $(w_n, w_{n+k}) \notin R_{L_d}$ for all $n \geq 0$ and all $k \geq 1$. Hence, R_{L_d} has infinite index, which implies that L_d is not accepted by any DFA. □

Problem 3.3 [DFAs]

Two states p, q of a DFA $A = (Q, \Sigma, \delta, q_0, F)$ are called *equivalent* if, for all $w \in \Sigma^*$, $\delta(p, w) \in F$ iff $\delta(q, w) \in F$. Find all pairs of equivalent states in the following four DFAs:

(a) $A_1 = (\{q_0, q_1, \ldots, q_7\}, \{a, b\}, \delta_1, q_0, \{q_0, q_4, q_6\})$, where δ_1 is described through the following table:

δ_1	q_0	q_1	q_2	q_3	q_4	q_5	q_6	q_7
a	q_0	q_1	q_2	q_3	q_6	q_5	q_4	q_7
b	q_5	q_3	q_7	q_2	q_1	q_1	q_2	q_0

(b) $A_2 = (\{A, B, C, D, E, F\}, \{a, b\}, \delta_2, F, \{F\})$, where δ_2 is described by the following table:

δ_2	A	B	C	D	E	F
a	A	B	C	D	E	F
b	F	A	D	B	C	E

(c) $A_3 = (\{q_1, q_2, \ldots, q_9\}, \{a, b\}, \delta_1, q_1, \{q_3, q_5, q_6\})$, where δ_3 is described through the following table:

δ_3	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
a	q_2	q_2	q_3	q_2	q_6	q_6	q_7	q_2	q_9
b	q_3	q_4	q_5	q_7	q_3	q_6	q_4	q_3	q_4

(d) $A_4 = (\{A, B, C, D, E, F, G, H\}, \{a, b\}, \delta_4, G, \{G\})$, where δ_4 is described by the following table:

δ_4	A	B	C	D	E	F	G	H
a	H	B	E	D	C	F	G	A
b	G	A	D	B	D	E	F	G

Solution.

(a) In A_1 states q_4 and q_6 are unreachable. They are not equivalent, as $\delta_1(q_4, b^5) = q_0 \in F_1$, while $\delta_1(q_6, b^5) = q_1 \notin F_1$. Further, $\delta_1(q_0, b^5) = q_7 \notin F_1$, $\delta_1(q_0, b^6) = q_0 \in F_1$, and $\delta_1(q_6, b^6) = q_3 \notin F_1$, which shows that q_0 is neither equivalent to q_4 nor to q_6. Finally, we have $q_5 \to^b q_1 \to^b q_3 \to^b q_2 \to^b q_7 \to^b q_0$, which is the only final state in this sequence. Hence, none of these states is equivalent to any other state. Thus, there are no equivalent states in A_1

(b) As $\delta_2(q, a) = q$ for all $q \in Q_2$, and as $F \to^b E \to^b C \to^b D \to^b B \to^b A \to^b F$, and F is the only final state, we see none of the states of A_2 are equivalent.

(c) In A_3, q_8 and q_9 are unreachable, but they are not equivalent, as $\delta_3(q_8, b) = q_3 \in F_3$, while $\delta_3(q_9, b) = q_4 \notin F_3$. Next, it is easily seen that the three final states q_3, q_5, q_6 are all equivalent. Further, q_2, q_4, q_7, q_9 are equivalent. Finally, q_1 and q_8 are equivalent.

(d) For A_4, the states A and H are equivalent, and the states C and E are equivalent. □

Problem 3.4 [Minimal DFA]

Use the algorithm "minimal automaton" to construct the minimal DFAs that are equivalent to the following DFAs:

(a) $B_1 = (\{q_1, q_2, \ldots, q_9\}, \{a, b\}, \delta_1, q_1, \{q_3, q_5, q_6\})$, where δ_1 is described through the following table:

δ_1	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9
a	q_2	q_2	q_3	q_2	q_6	q_6	q_7	q_2	q_9
b	q_3	q_4	q_5	q_7	q_3	q_6	q_4	q_3	q_4

(b) $B_2 = (\{A, B, C, D, E, F, G, H\}, \{a, b\}, \delta_2, G, \{G\})$, where δ_2 is described by the following table:

δ_2	A	B	C	D	E	F	G	H
a	H	B	E	D	C	F	G	A
b	G	A	D	B	D	E	F	G

(c) $B_3 = (\{q_0, q_1, \ldots, q_5\}, \{a, b\}, \delta_3, q_0, \{q_0\})$, where δ_3 is described through the following table:

δ_3	q_0	q_1	q_2	q_3	q_4	q_5
a	q_1	q_3	q_4	q_0	q_2	q_0
b	q_2	q_0	q_5	q_2	q_5	q_3

Solution.

(a) States q_8 and q_9 are not reachable. Hence, we only need to consider the subautomaton consisting of states $q_1, q_2, \ldots, q_7$. It has initial state q_1 and final states q_3, q_5, and q_6. Here we obtain the following table:

q_2	X					
q_3	X	X				
q_4	X		X			
q_5	X	X		X		
q_6	X	X		X		
q_7	X		X		X	X
	q_1	q_2	q_3	q_4	q_5	q_6

Thus, the resulting minimal DFA has initial state $p_1 = \{q_1\}$, final state $p_2 = \{q_3, q_5, q_6\}$, and state $p_3 = \{q_2, q_4, q_7\}$, and its transition function δ_3' is given by the following table:

δ_3'	p_1	p_2	p_3
a	p_3	p_2	p_3
b	p_2	p_2	p_3

□