Automata and Grammars

SS 2018

Assignment 1: Solutions to Selected Problems

Seminary: Thursday, March 1, 2018.

Problem 1.1 (a)[Words]

Let Σ be a finite alphabet. Prove that the operation of concatenation $\cdot : \Sigma^* \times \Sigma^* \to \Sigma^*$ is commutative if and only if Σ has cardinality one.

Solution. If $\Sigma = \{a\}$, then each word $x \in \Sigma^*$ can be written as $x = a^m$ for some $m \geq 0$. Hence, $xy = a^m a^n = a^{m+n} = a^{n+m} = a^n a^m = yx$ for all $x, y \in \Sigma^*$, which shows that in this situation the operation of concatenation is commutative.
Conversely, if $|\Sigma| \geq 2$, then let a and b be two different letters from Σ. Now consider the words $x = a$ and $y = b$. Then $xy = ab \neq ba = yx$, that is, in this case the operation of concatenation is not commutative. □

Problem 1.1 (b)[Words]

Show that, for two words $u, v \in \Sigma^*$, $uv = vu$ if and only if there exist a word $x \in \Sigma^*$ and integers $k, l \geq 1$ such that $u = x^k$ and $v = x^l$.

Hint: Use induction on $|u| + |v|$.

Solution. If $|u| + |v| \leq 2$, then the result obviously holds. Assume that the result has already been shown to hold for all words u, v satifying $|u| + |v| \leq n$, and assume now that $|u| + |v| = n + 1$. From the problem presented in the first seminary we see that $u = xy$, $v = (xy)^k x$, and $u = yx$ for some words $x, y \in \Sigma^*$ and an integer $k \geq 0$. Then $xy = u = yx$, and as $|x| + |y| \leq n$, the induction hypothesis yields that $x = r^m$ and $y = r^n$ for some word $r \in \Sigma^*$ and some integers $m, n \geq 0$. This, however, means that $u = xy = r^{m+n}$ and $v = (xy)^k x = r^{k \cdot (m+n)+m}$. □

Problem 1.2. [Regular Grammars]

Construct regular grammars for at least three of the following languages:

(a) $L_1 = \{ w \in \{a, b\}^* \mid |w|_a \text{ is divisible by } 2 \}$,
(b) $L_2 = \{ w \in \{a, b\}^* \mid |w|_a \text{ is divisible by 2 and by } 3 \}$,
(c) $L_3 = \{ w \in \{a, b\}^* \mid w = uabab \text{ for some word } u \in \{a, b\}^* \}$,
(d) $L_4 = \{ w \in \{a, b\}^* \mid w \text{ contains the factor } abab \}$,
(e) $L_5 = \{ w \in \{a, b\}^* \mid |w| = (3k + 1) \text{ for some } k \geq 0 \text{ or } w \text{ ends with } b \}$,
(f) $L_6 = \{ w \in \{a, b\}^* \mid \text{The first and the last letter of } w \text{ are identical} \}$.

Solution. (b) $G_b = (\{S, A, B, C, D, E\}, \{a, b\}, P_b, S)$, where P_b contains the following productions:

$$S \to bS, S \to aD, S \to \varepsilon,$$
$$A \to bA, A \to aE, B \to bB, B \to aC,$$
$$C \to bC, C \to aA, D \to bD, D \to aB,$$
$$E \to bE, E \to aS.$$

G_b is the product of two grammars G_1 and G_2, where $S = (S_1, S_2)$, $A = (S_1, A_2)$, $B = (S_1, B_2)$, $C = (A_1, S_2)$, $D = (A_1, A_2)$, and $E_(A_1, B_2)$, $P_1 = \{S_1 \to bS_1, S_1 \to aA_1, S_1 \to$

$\varepsilon, A_1 \to bA_1, A_1 \to aS_1\}$, and $P_2 = \{S_2 \to bS_2, S_2 \to aA_2, S_2 \to \varepsilon, A_2 \to bA_2, A_2 \to aB_2, B_2 \to bB_2, B_2 \to aS_2\}$. Then $L(G_1) = \{\, w \in \{a,b\}^* \mid |w|_a \equiv 0 \mod 2 \,\}$ and $L(G_2) = \{\, w \in \{a,b\}^* \mid |w|_a \equiv 0 \mod 3 \,\}$. Hence, $L(G_4) = L(G_1) \cap L(G_2) = L_4$.

(f) $G_f = (\{S, A, B\}, \{a, b\}, P_f, S)$, where P_f contains the following productions:

$$S \to a, S \to b, S \to aA, S \to bB, A \to aA, A \to bA, A \to a, B \to aB, B \to bB, B \to b.$$

Then $L(G_f) = L_6$.

Problem 1.3. [Regular Grammars]

Determine the languages that are generated by the following regular grammars:

(a) $G = (\{S, A, B\}, \{a, b\}, S, P)$, where P contains the following productions:

$$\begin{array}{lllllll} S & \to & aabA, & S & \to & bbaB, \\ A & \to & aabA, & A & \to & aab, \\ B & \to & bbaB, & B & \to & bba. \end{array}$$

(b) $G = (\{S, A, B\}, \{a, b\}, S, P)$, where P contains the following productions:

$$\begin{array}{llllll} S & \to & abA, & & & \\ A & \to & abA, & A & \to & abB, \\ B & \to & baB, & B & \to & ba. \end{array}$$

(c) $G = (\{S, A, B\}, \{a, b\}, S, P)$, where P contains the following productions:

$$\begin{array}{lllllllll} S & \to & Sb, & S & \to & Aa, & S & \to & \varepsilon, \\ A & \to & Ab, & A & \to & Ba, & & & \\ B & \to & Bb, & B & \to & Sa. & & & \end{array}$$

Solution. The following solutions can be shown easily:

$$\begin{array}{llll} (a) & L & = & \{\, (aab)^n \mid n \geq 2 \,\} \cup \{\, (bba)^n \mid n \geq 2 \,\}, \\ (b) & L & = & \{\, (ab)^m (ba)^n \mid m \geq 2, n \geq 1 \,\}, \\ (c) & L & = & \{\, w \in \{a,b\}^* \mid |w|_a \text{ is divisible by } 3 \,\}. \end{array}$$

(a) A typical derivation in G looks as follows:

$$S \to_G aabA \to_G^n (aab)^{n+1} A \to_G (aab)^{n+2},$$

where $n \geq 0$, or

$$S \to_G bbaB \to_G^n (bba)^{n+1} B \to_G (bba)^{n+2},$$

where $n \geq 0$. It follows that $L(G) = \{\, (aab)^n, (bba)^n \mid n \geq 2 \,\}$.

(b) A derivation in G looks as follows:

$$\begin{array}{lllllll} S & \to_G & abA & \to_G^m & (ab)^{m+1} A & \to_G & (ab)^{m+2} B \\ & \to_G^n & (ab)^{m+2} (ba)^n B & \to_G & (ab)^{m+2} (ba)^{n+1}, & & \end{array}$$

where $m, n \geq 0$. Hence, $L(G) = \{\, (ab)^m (ba)^n \mid m \geq 2, n \geq 1 \,\}$.

(c) Here we can show that

$$\begin{array}{rcl} L(G,S) & = & \{\, w \in \{a,b\}^* \mid |w|_a \equiv 0 \mod 3 \,\}, \\ L(G,A) & = & \{\, w \in \{a,b\}^* \mid |w|_a \equiv 2 \mod 3 \,\}, \text{ and} \\ L(G,B) & = & \{\, w \in \{a,b\}^* \mid |w|_a \equiv 1 \mod 3 \,\}. \end{array}$$

For the inclusions from left to right, this can be done by induction on the length of a derivation $X \to_G^n w \in \{a,b\}^*$, where $X \in \{S,A,B\}$. For the opposite direction this can be done by induction on the length of w.

□