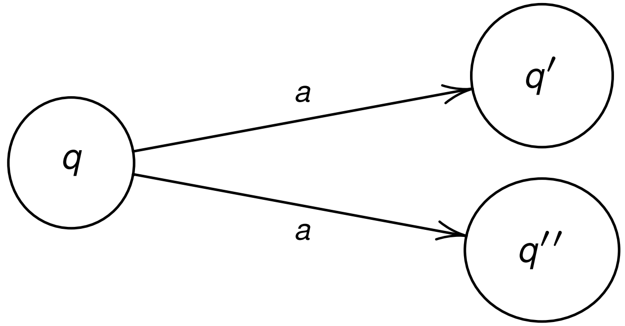
2.4 Nondeterministic Finite-State Automaton



Definition 2.14

A nondeterministic finite-state automaton (NFA) A is given through a 5-tuple $(Q, \Sigma, \delta, S, F)$, where

– *Q is a finite set of (internal) states,*
– *Σ is a finite (input) alphabet,*
– *$Q \cap \Sigma = \emptyset$,*
– *$S \subseteq Q$ is a set of initial states,*
– *$F \subseteq Q$ is a set of final states, and*
– *$\delta : Q \times \Sigma \to 2^Q$ is a transition relation.*

An NFA can be described by a state graph just like a DFA.

Definition 2.14 (cont.)

Let $A = (Q, \Sigma, \delta, S, F)$ be an NFA.
The transition relation δ is extended to a function $\hat{\delta}: \quad 2^Q \times \Sigma^* \to 2^Q$:

$$\begin{array}{lcl} \hat{\delta}(P, \varepsilon) & := & P \text{ for all } P \subseteq Q, \\ \hat{\delta}(P, ax) & := & \bigcup\limits_{q \in P} \hat{\delta}(\delta(q, a), x) \text{ for all } P \subseteq Q, a \in \Sigma, \text{ and } x \in \Sigma^*. \end{array}$$

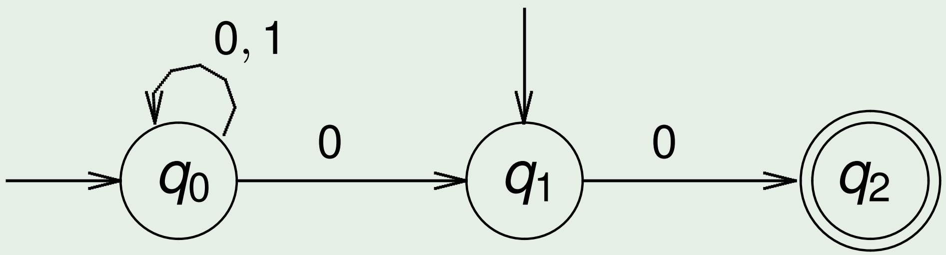
$L(A) := \{ x \in \Sigma^* \mid \hat{\delta}(S, x) \cap F \neq \emptyset \}$ is the language accepted by A.

Remark

$\hat{\delta}(S, a_1 a_2 \ldots a_n) \cap F \neq \emptyset$
iff
$\exists q_0, q_1, \ldots, q_n \in Q : q_0 \in S \ \wedge \ q_n \in F \ \wedge \ q_i \in \delta(q_{i-1}, a_i) \ \ (1 \leq i \leq n)$.
iff
$\exists q_0 \in S \, \exists q_n \in E$: In the state graph of A,
there is a directed path from q_0 to q_n with label $a_1 a_2 \ldots a_n$.

Example:

(a) A_1:



Let $x := 1100$. Then:

$S = \{q_0, q_1\} \xrightarrow{1} \{q_0\} \xrightarrow{1} \{q_0\} \xrightarrow{0} \{q_0, q_1\} \xrightarrow{0} \{q_0, q_1, q_2\}$

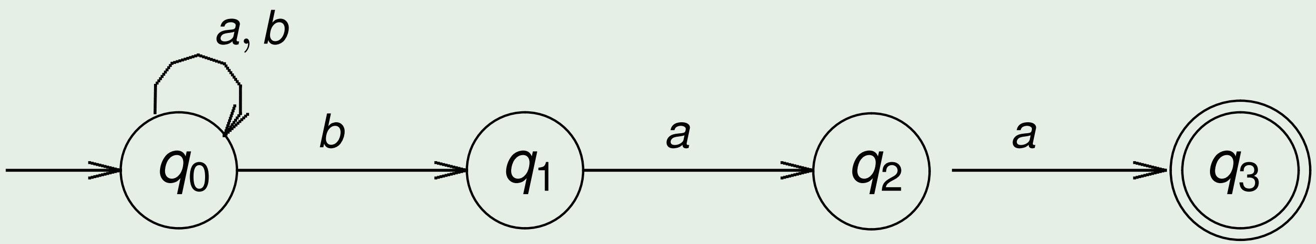
Hence, $x \in L(A_1)$.

In fact:

$$L(A_1) = \{ \ x \in \{0, 1\}^* \mid x = 0 \text{ or } \exists y \in \{0, 1\}^* : x = y00 \ \}.$$

Example (cont.):

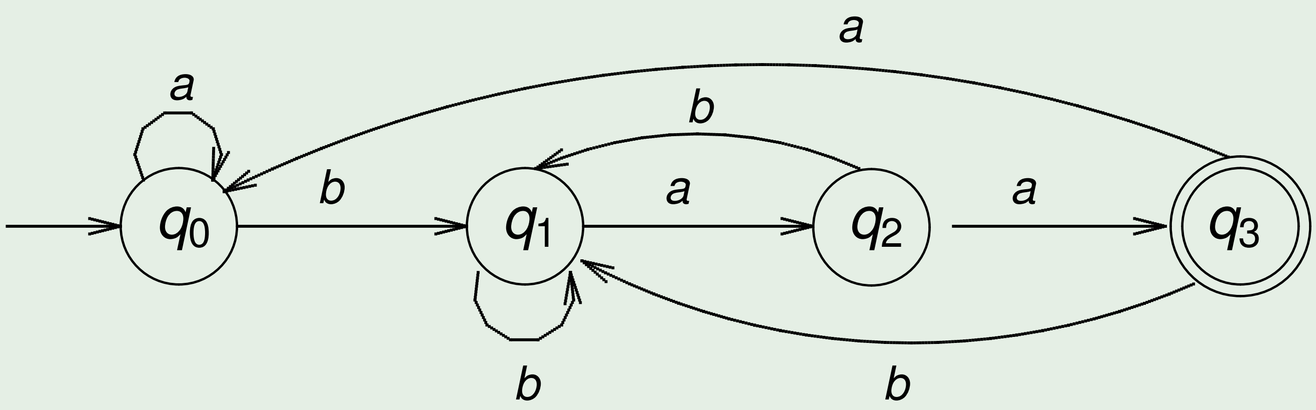
(b) A_2:



$L(A_2) = \{\ w \in \{a, b\}^* \mid w \text{ has suffix } baa\}.$

A DFA for this language:

A_3:



Theorem 2.15 (Rabin,Scott)

From an NFA A, a DFA B can be constructed such that $L(B) = L(A)$.

Proof.

Let $A = (Q, \Sigma, \delta_A, S, F)$ be an NFA.

Goal: A DFA $B = (P, \Sigma, \delta_B, p_0, G)$ such that $L(B) = L(A)$.

Power set construction:

$P := 2^Q$, $p_0 := S$,
$G := \{\, Q' \subseteq Q \mid Q' \cap F \neq \emptyset \,\}$,
$$\delta_B(Q', a) := \bigcup_{q \in Q'} \delta_A(q, a) = \hat{\delta}_A(Q', a)$$
for all $Q' \subseteq Q$ and all $a \in \Sigma$.

Proof of Theorem 2.15 (cont.)

Claim:

$L(B) = L(A)$.

Proof.

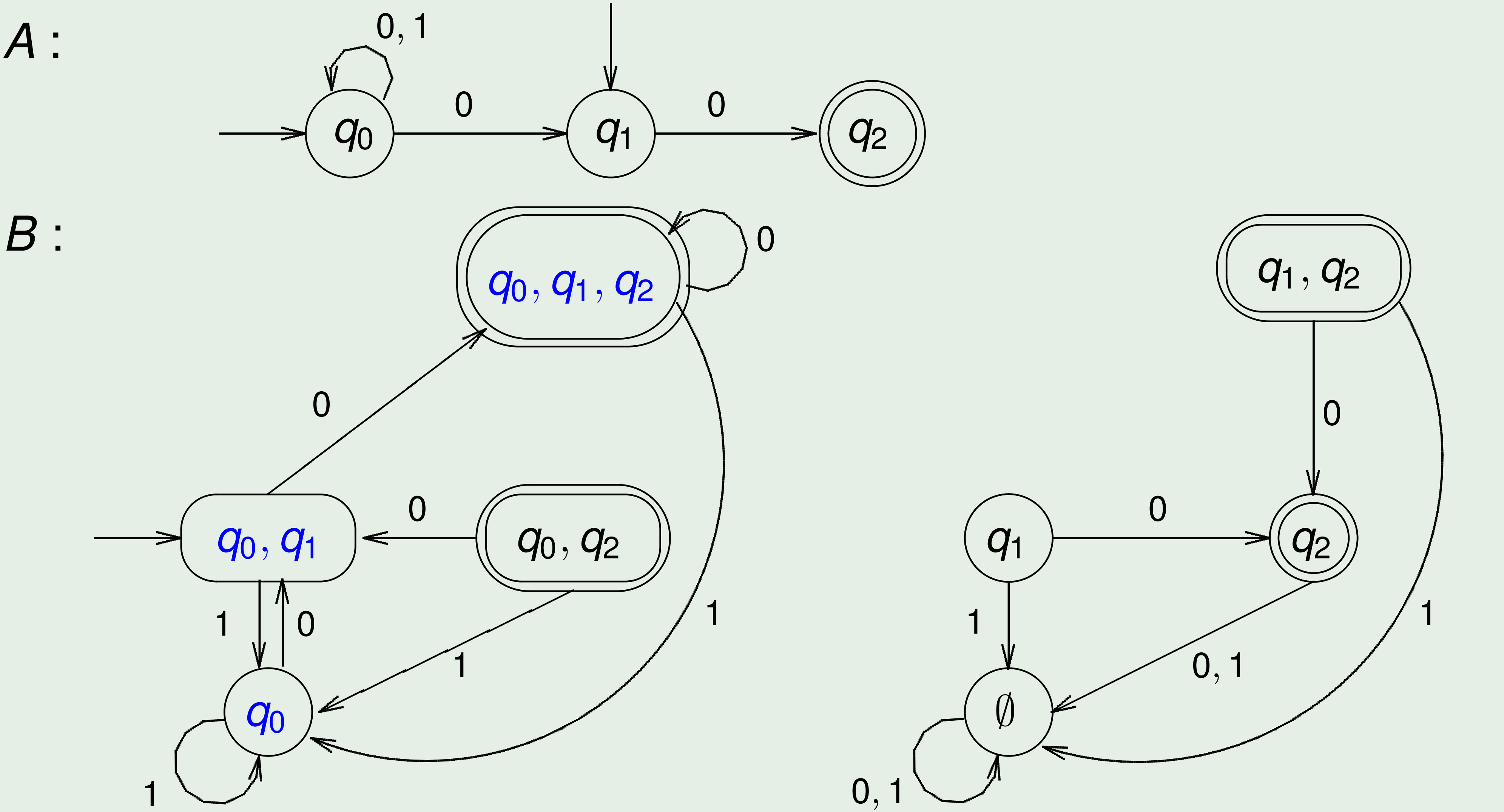
$\varepsilon \in L(A)$ iff $S \cap F \neq \emptyset$ iff $S \in G$ iff $\varepsilon \in L(B)$.

Now let $x = a_1 a_2 \ldots a_n \in \Sigma^+$.

Then: $x \in L(A)$ iff $\hat{\delta}_A(S, x) \cap F \neq \emptyset$
iff $\exists Q_1, Q_2, \ldots, Q_n \subseteq Q : \delta_B(S, a_1) = Q_1$,
$\delta_B(Q_1, a_2) = Q_2, \ldots, \delta_B(Q_{n-1}, a_n) = Q_n$ and $Q_n \cap F \neq \emptyset$
iff $\hat{\delta}_B(S, x) \in G$ iff $x \in L(B)$.

□□

Example (cont.):



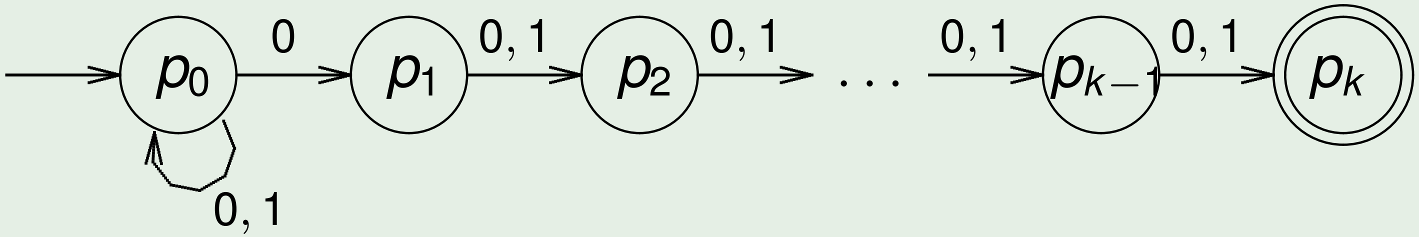
Remark

Only compute the subautomaton of B that is reachable from S!

Example:

$L_k := \{\, x = x_1 \ldots x_n \in \{0,1\}^* \mid |x| = n \geq k \text{ and } x_{n-k+1} = 0 \,\}$

An NFA A_k with $L(A_k) = L_k$:



Let $B = (Q, \{0,1\}, \delta, q_0, F)$ be a DFA such that $L(B) = L_k$.

Example (cont.):

Claim: $|Q| \geq 2^k$.

Proof.

Assume that $Q = \{q_0, q_1, \ldots, q_r\}$, where $r < 2^k - 1$.
Then: $\exists y_1, y_2 \in \{0,1\}^k : y_1 \neq y_2$ and $\hat{\delta}(q_0, y_1) = \hat{\delta}(q_0, y_2)$.

Let i be the leftmost position such that $y_1(i) \neq y_2(i)$,
w.l.o.g. $y_1 = u0v_1$ and $y_2 = u1v_2$.

Let $w \in \{0,1\}^{i-1}$ be arbitrary.

Then: $y_1w = u0v_1w$, $|v_1w| = k - i + i - 1 = k - 1$
$\leadsto y_1w \in L_k$
$y_2w = u1v_2w$, $|v_2w| = k - 1$
$\leadsto y_2w \notin L_k$.

$$\begin{aligned} \text{But:}\quad \hat{\delta}(q_0, y_1w) &= \hat{\delta}(\hat{\delta}(q_0, y_1), w) \\ &= \hat{\delta}(\hat{\delta}(q_0, y_2), w) \\ &= \hat{\delta}(q_0, y_2w), \text{ a contradiction!} \end{aligned}$$

$\square$