Agata Ciabattoni
Revantha Ramanayake
Heinrich Wansing

# Hypersequent and display calculi – a unified perspective

**Abstract.** This paper presents an overview of the methods of hypersequents and display sequents in the proof theory of non-classical logics. In contrast with existing surveys dedicated to hypersequent calculi or to display calculi, our aim is to provide a unified perspective on these two formalisms highlighting their differences and similarities and discussing applications and recent results connecting and comparing them.

*Keywords*: proof theory, hypersequent calculi, display calculi.

## 1. Introduction

It is well-known that Gerhard Gentzen's motivation to introduce the sequent calculus[1] arose from his investigations into natural deduction. As Gentzen [50, p. 289] explains:

> A closer investigation of the specific properties of the natural calculus has finally led me to a very general theorem which will be referred to below as the "*Hauptsatz.*" ... In order to be able to enunciate and prove the *Hauptsatz* in a convenient form, I had to provide a logical calculus especially suited to the purpose. For this the natural calculus proved unsuitable. For, although it already contains the properties essential to the validity of the *Hauptsatz*, it does so only with respect to its intuitionistic form.

Gentzen then developed the sequent calculi **LK** and **LJ** for first-order classical logic and first-order intuitionistic logic. A crucial rule in sequent calculi is the *cut-rule*. It may be thought of as corresponding to the transitivity of deduction or the introduction of lemmas into proofs. While (cut) is useful for shortening proofs and proving completeness relative to Hilbert calculi, it greatly hinders proof search and the extraction of useful information from

---

[1]When we refer to *the* sequent calculus, *the* hypersequent calculus or *the* display calculus, we refer to the respective proof-theoretic framework and not to any particular proof system within this framework.

proofs. A calculus not containing the cut-rule is said to be *cut-free*. Cut-free proof systems are helpful for proving various results for logical systems, including consistency, interpolation and decidability. To obtain a cut-free calculus, Gentzen proved the cut-elimination theorem (his Hauptsatz, main theorem) which shows how to eliminate instances of the cut-rule from a given derivation.

Whereas classical and intuitionistic logic were the central systems of formal logic at the time Gentzen completed his dissertation, the importance of non-classical logics other than intuitionistic logic emerged only in the second half of the 20th century. Logics intermediate between intuitionistic and classical logic, substructural logics, many-valued logics, paraconsistent logics, modal, epistemic, deontic, temporal, dynamic, non-monotonic and other logics nowadays are of focal interest, both from a purely theoretical perspective and from the point of view of applications in linguistics, philosophy and computer science. Subsequently cut-free sequent calculi have been presented for various non-classical logics.

However, there are many logics for which there is no known standard cut-free sequent calculus (e.g., the modal logic S5 [71] or first-order Gödel logic [4,26]) and some logics provably cannot have such a calculus [98]. Moreover, the sequent calculus suffers from a lack of modularity in the sense that it is often difficult to construct cut-free sequent systems for seemingly simple extensions of logics for which a cut-free sequent calculus exists. These considerations have prompted the search for new proof formalisms (see Section 5 for a brief overview of some of these). Most new formalisms are obtained as generalizations of Gentzen's sequent calculi.

A sequent has the form $X \Rightarrow Y$ where $X$ and $Y$ are comma-separated lists (or multisets) of formulae. The comma is the sole *structural* connective (aside from the sequent symbol $\Rightarrow$). It is not a connective of the logical language, and it is typically interpreted as conjunction in the antecedent and disjunction in the succedent. We consider two natural generalizations.

1. *Hypersequents* are obtained from ordinary sequents by the addition of a single new structural connective | used to separate the sequents.

2. *Display sequents* introduce (many) new structural connectives and explicit rules that govern the interaction between these connectives in order to obtain what is called the display property.

This paper presents an overview[2] of the methods of hypersequents and display sequents in the proof theory of non-classical logics. In contrast with

---

[2]We show only sketches of the proofs and provide references to the original papers.

existing surveys dedicated to hypersequent calculi (e.g. [4,6,76]) or to display calculi (e.g. [53, 103, 105, 108]) our aim is to provide a unified perspective of these two formalisms showing differences and similarities, and to discuss some of their applications as well as recent results connecting and comparing them.

## 2. Hypersequents

Hypersequent calculi have been introduced in [2] (and independently in [84]).

DEFINITION 1. *A hypersequent is a sequence, written as*

$$\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$$

*where, for all $i = 1, \ldots, n$, $\Gamma_i \Rightarrow \Pi_i$ is an ordinary sequent[3] called a* component *of the hypersequent.*

*A hypersequent is* single-conclusioned *if, for every $i = 1, \ldots, n$, $\Pi_i$ consists of at most one formula and is called* multiple-conclusioned *otherwise.*

The "|" symbol is usually interpreted as a disjunction. As explained in [4],

> although a hypersequent is certainly a more complex data structure than an ordinary sequent, it is not *much* more complicated, and goes in fact just one step further.

As with ordinary sequent calculi, the inference rules of hypersequent calculi consist of initial hypersequents (i.e., axioms), the cut-rule as well as logical and structural rules. These inference rules are usually presented as rule schemata. A concrete instance of a rule is obtained by instantiating the schematic variables with concrete formulae. In the case of hypersequents, the rule schema may also contain schematic context variables or *side hypersequents*, denoted by $G$ and $H$, representing (possibly empty) hypersequents. The logical and structural rules are divided into *internal* and *external rules*. The internal rules deal with formulae within one component of the conclusion. Examples of internal structural rules are the ordinary rules of weakening, exchange and contraction of the sequent calculus (augmented with the side hypersequents). Examples of external structural rules include external weakening (ew), external exchange[4] (ee) and external contraction (ec) (see

---

[3]In this paper we consider only commutative logics, hence $\Gamma_i$ and $\Pi_i$ are finite multisets of formulae ($\Pi_i$ is either a formula or empty in single-conclusioned calculi).

[4]The usual interpretation of the "|" symbol is the standard disjunction, and the (ee) rule is omitted by considering hypersequents to be *multisets* of sequents.

Table 1). These behave like weakening, exchange and contraction over whole components of hypersequents.

A *derivation* is defined in the usual way as a finite tree of hypersequents constructed using the axioms and rules. Let **H** be a hypersequent calculus. We write $\vdash_{\mathbf{H}} P$ to mean that **H** derives the formula $P$.

As an example, we now present the hypersequent version of the propositional sequent calculus **LJ** for intuitionistic logic, which we call **HIL**. Note that all hypersequents below are single-conclusioned ($\Pi$ and $\Pi'$ are schematic variables to be replaced by the empty set or a single formula).

*Axioms*                                       *Cut Rule*

$$\frac{G \mid \Gamma' \Rightarrow A \quad G' \mid A, \Gamma \Rightarrow \Pi}{G \mid G' \mid \Gamma, \Gamma' \Rightarrow \Pi} \; (cut)$$

$A \Rightarrow A \qquad \bot \Rightarrow A$

*External Structural Rules*

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi} \; (ew) \qquad\qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi} \; (ec) \qquad \frac{G \mid \Gamma' \Rightarrow \Pi' \mid \Gamma \Rightarrow \Pi \mid G'}{G \mid \Gamma \Rightarrow \Pi \mid \Gamma' \Rightarrow \Pi' \mid G'} \; (ee)$$

*Internal Structural Rules*

$$\frac{G \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma, A \Rightarrow \Pi} \; (w, l) \qquad\qquad \frac{G \mid \Gamma \Rightarrow}{G \mid \Gamma \Rightarrow \Pi} \; (w, r) \qquad \frac{G \mid \Gamma, A, A \Rightarrow \Pi}{G \mid \Gamma, A \Rightarrow \Pi} \; (c, l)$$

*Logical Rules*

$$\frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A {\to} B} \; (\to, r) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A \quad G \mid B, \Gamma \Rightarrow \Pi}{G \mid \Gamma, A {\to} B \Rightarrow \Pi} \; (\to, l)$$

$$\frac{G \mid \Gamma \Rightarrow A \quad G \mid \Gamma \Rightarrow B}{G \mid \Gamma \Rightarrow A \wedge B} \; (\wedge, r) \qquad\qquad \frac{G \mid \Gamma, A_i \Rightarrow \Pi}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow \Pi} \; (\wedge_i, l)_{i=1,2}$$

$$\frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2} \; (\vee_i, r)_{i=1,2} \qquad\qquad \frac{G \mid \Gamma, A \Rightarrow C \quad G \mid \Gamma, B \Rightarrow \Pi}{G \mid \Gamma, A \vee B \Rightarrow \Pi} \; (\vee, l)$$

$$\frac{G \mid \Gamma, A \Rightarrow}{G \mid \Gamma \Rightarrow \neg A} \; (\neg, r) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A}{G \mid \Gamma, \neg A \Rightarrow} \; (\neg, l)$$

Table 1. Hypersequent Calculus **HIL** for Intuitionistic Logic

As usual, the formula introduced by a logical rule in the conclusion is called the *principal formula*. For example, the principal formula of the rule

$(\to, r)$ and $(\neg, l)$ is, respectively, $A \to B$ and $\neg A$.

The "hyperlevel" of this calculus is in fact redundant since a hypersequent $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_k \Rightarrow \Pi_k$ is derivable in **HIL** if and only if $\Gamma_i \Rightarrow \Pi_i$ is derivable for some $i \in \{1, \ldots, k\}$. Indeed, any sequent calculus can be viewed trivially as a hypersequent calculus. The added expressive power of the latter is due to the possibility of defining new rules which act simultaneously on several components of one or more hypersequents. Table 2 displays some examples of such rules.

$$
\frac{G \mid \Gamma, \Gamma' \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow \Pi} \ (cl)
\qquad
\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma \Rightarrow \mid \Gamma' \Rightarrow} \ (lq)
\qquad
\frac{G \mid \Gamma_1, \Gamma_1' \Rightarrow \Pi \quad G \mid \Gamma_2, \Gamma_2' \Rightarrow \Pi'}{G \mid \Gamma_1, \Gamma_2' \Rightarrow \Pi \mid \Gamma_2, \Gamma_1' \Rightarrow \Pi'} \ (com)
$$

Table 2. Some external structural rules

The addition of these rules to the calculus **HIL** leads, respectively, to cut-free calculi for classical logic (**HCL**), Jankov's logic LQ (**HLQ**) and Gödel(-Dummett) logic (**HG**) [3,25]. As an aside note that classical logic can also be regained from **HIL** by removing the restriction that hypersequents are single-conclusioned.

EXAMPLE 2 ([3]). *A proof of* $(A \to B) \vee (B \to A)$ *in* **HG** *is as follows:*

$$
\frac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A} \ {\scriptstyle (com)}}{\Rightarrow A \to B \mid \ \Rightarrow B \to A} \ {\scriptstyle 2\mathrm{x}(\to,r)}}{\Rightarrow (A \to B) \vee (B \to A) \mid \ \Rightarrow (A \to B) \vee (B \to A)} \ {\scriptstyle 2\mathrm{x}(\vee_i,r)}}{\Rightarrow (A \to B) \vee (B \to A)} \ {\scriptstyle (ec)}
$$

Soundness and completeness for hypersequent calculi are often proved using the Hilbert system of the considered logic and the standard[5] interpretation below of hypersequents as formulae.

DEFINITION 3. *The* interpretation *of a* sequent $\Gamma \Rightarrow \Delta$, *denoted by* $\mathcal{I}(\Gamma \Rightarrow \Delta)$, *is defined as* $\bigwedge \Gamma \to \bigvee \Delta$, *where* $\bigwedge \Gamma$ *stands for the conjunction of the formulae in* $\Gamma$ *($\top$ if $\Gamma$ is empty), and* $\bigvee \Delta$ *for the disjunction of the formulae in* $\Delta$ *($\bot$ if $\Delta$ is empty). The* standard interpretation *of a* hypersequent $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ *is* $\mathcal{I}(\Gamma_1 \Rightarrow \Delta_1) \vee \cdots \vee \mathcal{I}(\Gamma_n \Rightarrow \Delta_n)$.

---

[5]For logics lacking some of the internal structural rules the interpretation is more involved, see [27].

We sketch below the proofs of soundness, completeness and cut-elimination for **HCL**, **HLQ** and **HG**. Recall that Hilbert calculi for the above logics are obtained via the addition of the following schemas to any Hilbert calculus for intuitionistic logic, see, e.g., [100] for the latter:

- axiom (schema) $(em)$ $A \vee \neg A$, for classical propositional logic CL
- axiom (schema) $(wem)$ $\neg A \vee \neg\neg A$, for Jankov logic LQ
- the linearity axiom (schema) $(lin)$ $(A{\to}B) \vee (B{\to}A)$, for Gödel logic G.

PROPOSITION 4 (Soundness and Completeness). *A formula $P$ is derivable in a Hilbert calculus for $\mathcal{L}$ if and only if $\vdash_{\mathbf{H}\mathcal{L}} P$, for $\mathcal{L} \in \{\mathrm{CL}, \mathrm{LQ}, \mathrm{G}\}$.*

PROOF. (Soundness). The interpretation of the axioms of **H$\mathcal{L}$** has the form $A{\to}A$ and $\bot{\to}A$. It easy to verify that these are derivable in the Hilbert calculus $h\mathcal{L}$ for $\mathcal{L}$. It remains to show for each rule (below left) of **H$\mathcal{L}$** that its interpretation below right is *admissible* in $h\mathcal{L}$, that is if $\mathcal{I}(S_1) \ldots \mathcal{I}(S_n)$ are derivable in $h\mathcal{L}$ then so is $\mathcal{I}(S_0)$:

$$\frac{S_1 \ldots S_n}{S_0} \,(r) \qquad\qquad \frac{\mathcal{I}(S_1) \ldots \mathcal{I}(S_n)}{\mathcal{I}(S_0)} \,\mathcal{I}(r)$$

For example, the interpretations of $(ew)$ and $(ee)$ are respectively:

$$\frac{\mathcal{I}(G)}{\mathcal{I}(G) \vee \mathcal{I}(\Gamma \Rightarrow \Pi)} \qquad \frac{\mathcal{I}(G) \vee \mathcal{I}(\Gamma' \Rightarrow \Pi') \vee \mathcal{I}(\Gamma \Rightarrow \Pi) \vee \mathcal{I}(G')}{\mathcal{I}(G) \vee \mathcal{I}(\Gamma \Rightarrow \Pi) \vee \mathcal{I}(\Gamma' \Rightarrow \Pi') \vee \mathcal{I}(G')}$$

Verifying the admissibility of these derivations in $h\mathcal{L}$ is straightforward. The case of the rules $(cl), (lq)$ and $(com)$ is handled by making use of the corresponding axiom schema.

(Completeness). First show that $\vdash_{\mathbf{H}\mathcal{L}} P_i$ for all schemas $P_i$ of $h\mathcal{L}$. This is straightforward. It remains to show that the rule of *modus ponens* in $h\mathcal{L}$ can be simulated in **H$\mathcal{L}$**. This amounts to deriving $\Rightarrow B$ from $\Rightarrow A$ and $\Rightarrow A \to B$. Since $A, A \to B \Rightarrow B$ is derivable in **H$\mathcal{L}$**, this follows from two applications of the cut-rule. ■

The completeness proof for **HCL**, **HLQ** and **HG** relies on the presence of the cut-rule. We show below that this rule is in fact eliminable from **H$\mathcal{L}$**-derivations, for $\mathcal{L} \in \{\mathrm{CL}, \mathrm{LQ}, \mathrm{G}\}$.

REMARK 5. *The rules $(cl), (lq)$ and $(com)$ are a reformulation of the axiom schemas $(em), (wem)$ and $(lin)$, respectively. An algorithm to automatically extract these rules out of the corresponding axioms was introduced in [26] and is sketched in Section 4.2.1.*

Cut-elimination is a central result to be proved for hypersequent calculi, and the proof usually proceeds similarly to the sequent calculus case.Various methods to remove applications of the cut-rule have been introduced starting with Avron's "history method" [2] – a rather complicated variant of Gentzen-style cut-elimination. We sketch below a proof that works for a large class of calculi obeying the syntactic conditions of substitutivity and reductivity first introduced in [22]. These conditions were inspired by the reformulation [90] for sequent calculi of Belnap's conditions C1-C8 for display calculi (see Section 3).

Below we present these conditions for single-conclusioned calculi and use them for **HCL**, **HLQ** and **HG**. Henceforth we write $X^n$ to mean $X, \ldots, X$ ($n$ occurrences). Now define the set $\mathrm{CUT}(G, H)$ consisting of all hypersequents that are obtained by applying (cut) between one component in $H$ and one or more components in $G$, in parallel, in all possible ways. More precisely, $\mathrm{CUT}(G, H)$ is the set of hypersequents obtained by saturating $\{G, H\}$ under the following two operations:

1. if $G = (G' \,|\, \Gamma_1, A^{\lambda_1} \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n, A^{\lambda_n} \Rightarrow \Delta_n)$ and $H = (H' \,|\, \Sigma \Rightarrow A)$, then, for all $0 \leq \mu_i \leq \lambda_i$ and $i = 1, \ldots, n$ it is the case that $\mathrm{CUT}(G, H) \ni$

$$G' \,|\, H' \,|\, \Gamma_1, \Sigma^{\mu_1}, A^{\lambda_1 - \mu_1} \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n, \Sigma^{\mu_n}, A^{\lambda_n - \mu_n} \Rightarrow \Delta_n$$

2. if $G = (G' \,|\, \Gamma_1 \Rightarrow A \,|\, \ldots \,|\, \Gamma_n \Rightarrow A)$ and $H = (H' \,|\, \Sigma, A \Rightarrow \Pi)$ then it is the case that $\mathrm{CUT}(G, H) \ni$

$$G' \,|\, H' \,|\, \Gamma_1, \Sigma \Rightarrow \Pi \,|\, \ldots \,|\, \Gamma_n, \Sigma \Rightarrow \Pi$$

DEFINITION 6. *A hypersequent rule* $(r)$ *is* substitutive *if for any:*

- *instance* $\dfrac{G_1 \ \ldots \ G_n}{G}$ *of* $(r)$;
- *single-conclusioned hypersequent* $H$;
- $G' \in \mathrm{CUT}(G, H)$ *(with the condition that if* $(r)$ *is a logical rule then* $G'$

*contains its principal formula) there exist* $G'_i \in \mathrm{CUT}(G_i, H)$ *for* $i = 1, \ldots, n$
*such that* $\dfrac{G'_1 \ \ldots \ G'_n}{G'}$ *is an instance of* $(r)$.

Intuitively, substitutivity ensures that (parallel) cuts over formulae that are not principal in the rule can be shifted upwards over the premises.

For the next definition we assume that logical rules introduce only one connective and are divided into left and right rules according to whether this connective is on the left or on the right hand side of the turnstile.

DEFINITION 7. *The logical rules for any n-ary connective $\star$ are* reductive *if for all instances of left and right rules for $\star$:*

$$\frac{G\,|\,S_1 \quad \ldots \quad G\,|\,S_l}{G\,|\,\Gamma, \star(A_1,\ldots,A_n) \Rightarrow \Pi} \qquad \frac{G'\,|\,S'_1 \quad \ldots \quad G'\,|\,S'_k}{G'\,|\,\Sigma \Rightarrow \star(A_1,\ldots,A_n)}$$

$G'\,|\,G\,|\,\Gamma, \Sigma \Rightarrow \Pi$ *is derivable from* $(G\,|\,S_1),\ldots(G\,|\,S_l),(G'\,|\,S'_1),\ldots,(G'\,|\,S'_k)$ *using only* (cut) *with cut-formulae from* $A_1,\ldots,A_n$.

Intuitively, reductivity ensures that when both premises introduce the cut-formula, we can replace the original cut with cuts on its subformulae. The sequent version of reductivity is called coherence in [5].

THEOREM 8 ([30,31,76]). *A hypersequent calculus consisting of (i) the identity axiom $A \Rightarrow A$ and possibly the usual axioms for constants, (ii) the rules* (cut), (ew), (ee) *and* (ec), *(iii) a set of substitutive and reductive logical rules, and (iv) a set of substitutive structural rules admits cut-elimination.*

PROOF. A Gentzen-style proof[6] of this theorem proceeds by eliminating a *topmost cut* in a derivation by primary induction on the complexity of the cut-formula and secondary induction on the sum of the lengths of its left and right derivations. In his original proof Gentzen [50] encountered the following problem: if the cut-formula is derived by contraction $((c,l)$ or $(c,r)$ in Table 1), the permutation of cut with contraction does not necessarily move the cut up in the derivation. His solution was to introduce the multicut-rule – a derivable generalization of cut – that, e.g., for **LJ** has the following form:

$$\frac{\Gamma \Rightarrow A \quad \Gamma', A^n \Rightarrow B}{\Gamma, \Gamma' \Rightarrow B} \ (mcut)$$

i.e., we cut the single formula $A$ with possibly multiple occurrences. In the multiple-conclusion calculus **LK** the multicut-rule removes possibly many formulae in both premises. In hypersequent calculi a similar problem arises when permuting cut with the external contraction rule $((ec)$ in Table 1). In analogy with Gentzen's solution, a solution is to consider suitable (derivable) generalizations of the multicut-rule, that in our case cut one component in $H$ with possibly many components in $G$.

The cut-elimination proceeds by shifting a topmost cut upwards in a specific order: first along the premise in which the cut-formula appears on the right (equivalently: left). If the cut-formula is principal in that premise,

---

[6] An alternative proof that makes no use of the multicut-rule, along the line of Belnap's cut-elimination theorem (Theorem 22), is contained in [21].

shift the cut upwards over the other premise. Replace by cuts on formulae with smaller complexity if the cut-formula is principal in both premises. Substitutivity ensures that all the rules allow multicuts to be shifted upwards in derivations until the cut-formula is principal in both premises. Reductivity ensures that cuts can be replaced by (smaller) cuts on subformulae. ∎

COROLLARY 9. *Cut-elimination holds for* **HCL**, **HLQ** *and* **HG**.

PROOF. The rules of **HIL** are clearly reductive and substitutive. It is easy to check that $(cl), (lq)$ and $(com)$ are substitutive. The claim follows from Theorem 8. ∎

## 2.1. Modal logics

Hypersequent calculi were successfully used to define cut-free calculi for various modal logics [4,31,57,65,69,73,81,84,91]. For example, the hypersequent calculus for S5 due to Avron [4] consists of the hypersequent version of the sequent calculus for S4 (see, e.g., [100]) with the following additional rule ("modalized splitting rule")

$$\frac{G \,|\, \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}{G \,|\, \Box\Gamma_1 \Rightarrow \Box\Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2} \ (MS)$$

We call the resulting calculus **HS5**. Note that this calculus is built on the hypersequent calculus for classical logic that is obtained by considering the hypersequent version of **LK**. Alternative hypersequent calculi for S5 can be found, e.g., in [81,84].

DEFINITION 10. *The* interpretation *of the* **HS5***-hypersequent* $\Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$ *is the formula* $\Box\mathcal{I}(\Gamma_1 \Rightarrow \Delta_1) \vee \cdots \vee \Box\mathcal{I}(\Gamma_1 \Rightarrow \Delta_1)$, *where* $\mathcal{I}(\Gamma_i \Rightarrow \Delta_i)$ *is as in Definition 3.*

PROPOSITION 11 ([4]). *A formula A is valid in S5 if and only if* $\vdash_{\textbf{HS5}} A$.

PROPOSITION 12. *Cut-elimination holds for* **HS5**.

PROOF. Observe that the $(MS)$ rule is not substitutive (and neither are the modal rules for S4). Hence cut-elimination needs a different proof method. The Gentzen-style method presented in [4] makes use of the complicated multicut-rule below, that allows parallel cuts between many components and many cut-formulae in both premises: If $H \,|\, \Gamma_1 \Rightarrow \Delta_1, A^{n_1} \,|\, \ldots \,|\, \Gamma_p \Rightarrow \Delta_p, A^{n_p}$ and $G \,|\, \Sigma_1, A^{m_1} \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_k, A^{m_k} \Rightarrow \Pi_k$ are cut-free provable in **HS5**, then so are

1. $H \,|\, G \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_p \Rightarrow \Delta_p \,|\, \Sigma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Sigma_k \Rightarrow \Pi_k$, if $A = \Box B$

2. $H \,|\, G \,|\, \Gamma_1, \ldots, \Gamma_p \Rightarrow \Delta_1, \ldots, \Delta_p$, otherwise

It is not difficult, though tedious, to check that the inductive hypothesis above works in all cases. ■

The hypersequent calculi we have seen thus far were obtained by consideration of the Hilbert calculi for the logics of interest. A different approach is explored, e.g., in [69] where hypersequent calculi for a large class of modal logics are obtained by transforming their *frame conditions* into suitable rules. The approach in [69] applies to a variety of modal logics including KT, KD, S5, K4D, K4.2, K4.3, KBD, KBT, as well as logics characterized by frames of bounded cardinality, bounded width and bounded top width. Soundness and cut-free completeness (or completeness with analytic cuts, in the case of symmetric Kripke frames) are also provided. The method works for frame conditions having a certain shape (*simple frame properties*) defining a proper subclass of the geometric formulae handled in [78] using labelled calculi (see Section 5).

## 2.2.  Łukasiewicz logic

Thus far we have extended a base hypersequent version of a sequent calculus by the addition of structural or modal rules. There also exist hypersequent calculi having logical rules with no obvious sequent counterpart. Examples include the hypersequent calculi for the infinite-valued product [74] and Łukasiewicz [75] logic, or for the intermediate logic $\mathrm{BD}_2$ [28] which is semantically characterized by Kripke frames with depth at most 2.

We describe below the hypersequent calculus for Łukasiewicz logic Ł. The semantics of Ł was introduced in the 1920's and the logic is now recognized as one of the main formalizations of fuzzy logic [62,76]. An *interpretation $v$* for Ł is a function from the set of formulae[7] to $[0,1]$ such that $v(\bot) = 0$ and $v(A \to B) = \min(1, 1 - v(A) + v(B))$. As usual, a formula is valid in Łukasiewicz logic if it takes value 1 under all interpretations.

A Hilbert system for Łukasiewicz logic is obtained by the addition of the schema $((A \to B) \to B) \to ((B \to A) \to A)$ to any axiomatization of the multiplicative additive fragment of affine linear logic aMALL (that is MALL with weakening rules). Ideally we would like to extend the (hypersequent version of the) calculus for aMALL with further structural rules that capture the additional axiom schema. However it is known that no such

---

[7] $\{\bot, \to\}$ is a functionally complete set of basic connectives for Łukasiewicz logic.

structural hypersequent rule exists (see, e.g., Example 7.4 of [26]). To define a hypersequent calculus for Łukasiewicz logic, Metcalfe *et al.* [75, 76] used a different approach based on hypersequents that have no interpretation in the language of the logic. The resulting (multiple-conclusion) calculus **HL** consists of the following axioms and rules:

Axioms

$$(ID) \quad A \Rightarrow A \qquad\qquad (\Lambda) \quad \Rightarrow \qquad\qquad (\bot) \quad \bot \Rightarrow A$$

Internal structural rules

$$\frac{G \,|\, \Gamma \Rightarrow \Delta}{G \,|\, \Gamma, A \Rightarrow \Delta} \ (WL) \qquad\qquad \frac{G \,|\, \Gamma_1 \Rightarrow \Delta_1 \quad G \,|\, \Gamma_2 \Rightarrow \Delta_2}{G \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \ (M)$$

External structural rules

$$\frac{G}{G \,|\, \Gamma \Rightarrow \Delta} \ (ew) \qquad\qquad \frac{G \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma \Rightarrow \Delta}{G \,|\, \Gamma \Rightarrow \Delta} \ (ec)$$

$$\frac{G \,|\, \Gamma' \Rightarrow \Delta' \,|\, \Gamma \Rightarrow \Delta \,|\, G'}{G \,|\, \Gamma \Rightarrow \Delta \,|\, \Gamma' \Rightarrow \Delta' \,|\, G'} \ (ee) \qquad \frac{G \,|\, \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{G \,|\, \Gamma_1 \Rightarrow \Delta_1 \,|\, \Gamma_2 \Rightarrow \Delta_2} \ (S)$$

Logical rules

$$\frac{G \,|\, \Gamma, B \Rightarrow A, \Delta \,|\, \Gamma \Rightarrow \Delta}{G \,|\, \Gamma, A \to B \Rightarrow \Delta} \ (\to, l) \qquad \frac{G \,|\, \Gamma \Rightarrow \Delta \quad G \,|\, \Gamma, A \Rightarrow B, \Delta}{G \,|\, \Gamma \Rightarrow A \to B, \Delta} \ (\to, r)$$

EXAMPLE 13. *Here is a proof in* **HL** *of the characteristic axiom schema for Łukasiewicz logic:*

$$\frac{\begin{array}{c} \cfrac{\cfrac{\cfrac{B \Rightarrow B \quad A \Rightarrow A}{B, A \Rightarrow A, B} \ (M)}{B, A \Rightarrow A, B \,|\, B \Rightarrow A} \ (EW)}{B, B \to A \Rightarrow A} \ (\to, l) \quad \cfrac{\cfrac{\cfrac{B \Rightarrow B \quad A \Rightarrow A}{B, A \Rightarrow A, B} \ (M)}{B, B \to A, A \Rightarrow A, B} \ (WL)}{B, B \to A \Rightarrow A, A \to B} \ (\to, r)}{} \end{array}}{}$$

$$\frac{\cfrac{\Rightarrow}{(A \to B) \to B \Rightarrow} \ (WL) \qquad \cfrac{\cfrac{B, B \to A \Rightarrow A, A \to B \quad B, B \to A \Rightarrow A, A \to B \,|\, B \to A \Rightarrow A}{(A \to B) \to B, B \to A \Rightarrow A} \ (\to, l)}{(A \to B) \to B \Rightarrow (B \to A) \to A} \ (\to, r)}{\Rightarrow ((A \to B) \to B) \to ((B \to A) \to A)} \ (\to, r)$$

The following semantic criterion for validity of **HL** hypersequents is defined in [75]: we say that $\Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n$ is *valid*, in symbols $\models^*_{\text{Ł}} \Gamma_1 \Rightarrow \Delta_1 \,|\, \dots \,|\, \Gamma_n \Rightarrow \Delta_n$, iff for all interpretations $v$ for Łukasiewicz logic there exists $i$ such that $1 + \sum_{A \in \Gamma_i} [v(A) - 1] \leq 1 + \sum_{B \in \Delta_i} [v(B) - 1]$.

We emphasize that for *formulae*, this interpretation gives the usual notion of validity, i.e., a formula $A$ is valid in Łukasiewicz logic iff $\models^*_{\text{Ł}} \Rightarrow A$.

Note that under this interpretation, axiom $(\Lambda)$ simply means $1 \leq 1$.

PROPOSITION 14. *If $\vdash_{\mathbf{HL}} A$ then the formula $A$ is valid in Łukasiewicz logic.*

PROOF. The proof is by induction of the length on the derivation of $A$ in **HL**. See [75, 76] for details. ∎

The converse of this result (completeness) was proved in [75] using a semantic argument. The proof also shows that the cut-rule is admissible. A constructive proof that this rule can be removed from **HL** derivations is delicate. Note indeed that the logical rules of **HL** are not reductive and none of the general cut-elimination proofs in [22, 26, 31, 34, 76] work for **HL**. The special cut-elimination method, sketched below, was first introduced in [29].

PROPOSITION 15. **HL** *admits cut-elimination.*

PROOF. The cut-elimination proof makes use of the following two cut-rules:

$$\frac{G\,|\,\Gamma, A \Rightarrow \Delta \quad G'\,|\,\Pi \Rightarrow A, \Sigma}{G\,|\,G'\,|\,\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (cut) \qquad \frac{G\,|\,\Gamma, A \Rightarrow A, \Delta}{G\,|\,\Gamma \Rightarrow \Delta}\ (gencut)$$

First observe that the above cut-rules are *interderivable* in **HL**:

$$\frac{\dfrac{\dfrac{\dfrac{\Gamma, A \Rightarrow A, \Delta}{\Gamma \Rightarrow \Delta\,|\,\Gamma, A \Rightarrow A, \Delta}\ (EW)}{\Gamma, A \to A \Rightarrow \Delta}\ (\to,l) \quad \dfrac{\Rightarrow\ A \Rightarrow A}{\Rightarrow A \to A}\ (\to,r)}{\Gamma \Rightarrow \Delta}\ (cut)}{} \qquad \frac{\dfrac{\Gamma, A \Rightarrow \Delta \quad \Pi \Rightarrow A, \Sigma}{\Gamma, \Pi, A \Rightarrow A, \Delta, \Sigma}\ (M)}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ (gencut)$$

Elimination [29] is proved for $\mathbf{HL} + (gencut) + (cut)$-on *atomic* cut-formulae. Cut-elimination for $\mathbf{HL} + (cut)$ is then immediate due to the interderivability of $(cut)$ and $(gencut)$. To prove elimination, it is first shown that applications of $(cut)$ on atomic cut-formulae can be eliminated. This fact is then used to show the elimination of $(gencut)$ as follows: it is proved that the logical rules of **HL** are all *invertible* (i.e., that the premises of each rule are derivable if the conclusion is derivable); this allows us to reduce all applications of $(gencut)$ to applications of $(gencut)$ on atomic formulae. Finally, applications of $(gencut)$ on atomic formulae $p$ are eliminated. Note that when $(gencut)$ is preceded by an application of rule $(M)$ such that the atomic gencut-formula $p$ occurs in both premises of $(M)$, then $(gencut)$ cannot be shifted upward; in this case the identical conclusion can be obtained by applying $(cut)$ on the premises of $(M)$ with cut-formula $p$. Since this is a cut on the atomic cut-formula $p$ it can be eliminated. ∎

## 2.3. First-order quantifiers

A cut-free hypersequent calculus for first-order intuitionistic logic is obtained by adding to **HIL** the following natural rules for quantifiers [12, 22, 76]:

$$\frac{G \mid A(t), \Gamma \Rightarrow \Pi}{G \mid (\forall x) A(x), \Gamma \Rightarrow \Pi} \ (\forall, l) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A(a)}{G \mid \Gamma \Rightarrow (\forall x) A(x)} \ (\forall, r)$$

$$\frac{G \mid A(a), \Gamma \Rightarrow \Pi}{G \mid (\exists x) A(x), \Gamma \Rightarrow \Pi} \ (\exists, l) \qquad\qquad \frac{G \mid \Gamma \Rightarrow A(t)}{G \mid \Gamma \Rightarrow (\exists x) A(x)} \ (\exists, r)$$

where the rules $(\forall, r)$, $(\exists, l)$ must obey the eigenvariable condition: the free variable $a$ must not occur in the lower *hypersequent*. Henceforth we refer to this calculus as **HIL**$^{fo}$.

Soundness is immediate because $\Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_k \Rightarrow \Pi_k$ is derivable in **HIL**$^{fo}$ if and only if $\Gamma_i \Rightarrow \Pi_i$ is derivable for some $i \in \{1, \ldots, k\}$. Completeness and cut-elimination are also easy. However, the addition of new external structural rules to **HIL**$^{fo}$ may have side-effects. To see this, let **HG**$^{fo}$ be the calculus obtained by extending **HIL**$^{fo}$ with the communication rule (*com*) (see Table 2). **HG**$^{fo}$ is a calculus for first-order Gödel logic $G^{fo}$ (also known as Intuitionistic Fuzzy Logic [97]), axiomatized by adding to any Hilbert axiom for first-order intuitionistic logic not only the schema (*lin*) $(A{\rightarrow}B) \vee (B{\rightarrow}A)$ but also the quantifier shift[8] schema (*cd*) $\forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)$, where $x$ is not free in $B$.

REMARK 16. *The eigenvariable condition in $(\forall, r)$ and $(\exists, l)$ cannot be weakened to apply to just one component (i.e., "a must not occur in the instantiation of $\Gamma, \Pi, A$") as otherwise, e.g., in* **HG**$^{fo}$ *using (com) we could derive $\exists x A(x) \Rightarrow \forall x A(x)$ for each formula $A$.*

PROPOSITION 17 (Soundness and Completeness [12]). *A formula $P$ is derivable in a Hilbert calculus for $G^{fo}$ if and only if $\vdash_{\mathbf{HG}^{fo}} P$.*

PROOF. (Soundness) By Proposition 4 we only have to show the soundness of the quantifier rules. This is easy in the case of $(\forall, l)$ and $(\exists, r)$. For $(\forall, r)$ we may argue as follows: If $\mathcal{I}(G) \vee (\bigwedge \Gamma {\rightarrow} A(a))$ is derivable in the Hilbert calculus for $G^{fo}$, so is $\forall x(\mathcal{I}(G) \vee (\bigwedge \Gamma {\rightarrow} A(x)))$. Since $a$ did not occur in $G$ or in $\Gamma, A$ by the eigenvariable condition, we may now assume that $x$ does not either. Using (*cd*) we obtain $\mathcal{I}(G) \vee \forall x(\bigwedge \Gamma {\rightarrow} A(x))$. Since $\forall x(\bigwedge \Gamma {\rightarrow} A(x)) {\rightarrow} (\bigwedge \Gamma {\rightarrow} \forall x A(x))$ is derivable in first-order Gödel logic we

---

[8]$G^{fo}$ is indeed both a many-valued and an intermediate logic semantically characterized by the class of all rooted *linearly ordered* Kripke models with *constant domains*.

can derive $\mathcal{I}(G) \vee (\bigwedge \Gamma {\rightarrow} \forall x A(x))$ as required. The soundness of $(\exists, l)$ can be proved in a similar way.

(Completeness) $(lin)$ and the schemata of first-order intuitionistic logic are easily derivable. A derivation of $(cd)$ is the following:

$$
\dfrac{
\dfrac{
\dfrac{
A(a) \Rightarrow A(a) \qquad
\dfrac{
\dfrac{
\dfrac{
\dfrac{A(a) \Rightarrow A(a) \qquad B \Rightarrow B}{B \Rightarrow A(a) \mid A(a) \Rightarrow B} {\scriptstyle (com)} \qquad B \Rightarrow B
}{A(a) \vee B \Rightarrow A(a) \mid A(a) \vee B \Rightarrow B} {\scriptstyle 2x(\vee,l)+(ew)'s}
}{\forall x(A(x) \vee B) \Rightarrow A(a) \mid \forall x(A(x) \vee B) \Rightarrow B} {\scriptstyle 2x(\forall,l)}
}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \mid \forall x(A(x) \vee B) \Rightarrow B} {\scriptstyle (\forall, r)}
}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B \mid \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B} {\scriptstyle 2x(\vee, r)}
}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B} {\scriptstyle (ec)}
}{\Rightarrow \forall x(A(x) \vee B) \rightarrow (\forall x A(x) \vee B)} {\scriptstyle (\rightarrow, r)}
$$

$\blacksquare$

The proof that $G^{fo}$ admits cut-elimination[9] is easy as the propositional proof in Theorem 8 easily extends to quantifiers, see e.g. [30].

To define a cut-free hypersequent calculus for $G^{fo}$ without $(cd)$ (this logic is known in the literature as Corsi's logic [38] or Dummett's logic quantified) Tiu [99] introduced a variant of hypersequents in which eigenvariables form an explicit part of the syntactic structure. The quantifier rules in this calculus differ from the rules above and are based on the new structure of hypersequents, whose components have the form $\Sigma; \Gamma \Rightarrow \Pi$, where $\Gamma \Rightarrow \Pi$ is a ordinary single-conclusioned sequent and $\Sigma$ is a set of eigenvariables. These hypersequents have no formula-interpretation in the logic (see Section 3.4 for a similar phenomenon in first-order display calculi) and do not easily generalize to other logics.

## 3. Display calculi

Belnap's [15] display logic generalizes Gentzen's sequent calculus by supplementing the structural connective (,) and the turnstile ($\Rightarrow$) with a host of new structural connectives and rules manipulating these connectives. Since it is not really a logic but instead a framework for presenting proof-systems for logics, it makes sense to use the term "display calculi". The framework

---

[9]Note that the original proof [12] does not work as it incorrectly bounds the number of applications of $(ec)$ in cut-free derivations.

is a powerful formalism that has been used to capture a variety of different logics including resource-oriented logics [18], substructural [53], constructive [109], as well as modal and temporal logics [41,68,103,105]. The beauty of display calculi lies to a large extent in a general cut-elimination theorem for all calculi obeying eight easily verifiable syntactic conditions.

A *structure* is built from formulae using the structural connectives. We write $X, Y, \ldots$ for structures, $A, B, \ldots$ for formulae and $p, q, \ldots$ for propositional variables.

DEFINITION 18. *A* (display) sequent $X \Rightarrow Y$ *is built from structures* $X$ *and* $Y$. *The* $X$ *(resp.* $Y$*) is called an* a-part *structure (*s-part *structure).*

A characteristic feature of the display calculus is the *display property*, which states that every occurrence of a substructure in a sequent can be written (*displayed*) as the entire antecedent or succedent (but not both). Rules enabling the display property are called *display rules* or *display equivalences*. These rules are invertible and hence a sequent can be identified with the class of its display-equivalent sequents.

EXAMPLE 19. *Consider structures given by the following grammar.*

$$X ::= A \text{ is a formula} \mid \mathbf{I} \mid (X \circ X) \mid *X \mid \bullet X$$

*Then the sequent* $(\bullet p) \circ (*q) \Rightarrow p \circ (\bullet \bullet r)$ *contains the following substructures (note that* $p$ *appears as a substructure in the antecedent and the succedent and that some outermost brackets have been inserted for better readability).*

$$(\bullet p) \circ (*q) \quad \bullet p \quad p \quad *q \quad q \quad p \circ (\bullet \bullet r) \quad \bullet \bullet r \quad \bullet r \quad r$$

*The structures given by the above grammar come from the display calculus* $\delta Kt$ *[68, 103] described in Section 3.1. Using the display rules for this calculus (Table 3) we demonstrate how to display the substructures* $q$ *(below left) and* $\bullet r$ *(below right) in the sequent* $(\bullet p) \circ (*q) \Rightarrow p \circ (\bullet \bullet r)$.

$$\frac{\dfrac{(\bullet p) \circ (*q) \Rightarrow p \circ (\bullet \bullet r)}{*q \Rightarrow *(\bullet p) \circ (p \circ (\bullet \bullet r))}}{*(*(\bullet p) \circ (p \circ (\bullet \bullet r))) \Rightarrow q} \qquad \frac{\dfrac{(\bullet p) \circ (*q) \Rightarrow p \circ (\bullet \bullet r)}{(*p) \circ ((\bullet p) \circ (*q)) \Rightarrow \bullet \bullet r}}{\bullet((*p) \circ ((\bullet p) \circ (*q))) \Rightarrow \bullet r}$$

A *structural rule* is constructed from schematic structure variables and structure constants using the structural connectives. Typically, a display calculus consists of some structural rules (these include the display rules), *logical rules* introducing the logical connectives, initial sequents (axioms) and the cut-rule (below). Here the formula $A$ is called the *cut-formula*:

$X \circ Y \Rightarrow Z \,\|\, X \Rightarrow Z \circ *Y$         $X \circ Y \Rightarrow Z \,\|\, Y \Rightarrow *X \circ Z$

$X \Rightarrow Y \circ Z \,\|\, X \circ *Z \Rightarrow Y$         $X \Rightarrow Y \circ Z \,\|\, *Y \circ X \Rightarrow Z$

$*X \Rightarrow Y \,\|\, *Y \Rightarrow X$                 $X \Rightarrow *Y \,\|\, Y \Rightarrow *X$

$**X \Rightarrow Y \,\|\, X \Rightarrow Y$                 $X \Rightarrow **Y \,\|\, X \Rightarrow Y$

A further display rule for $\delta$Kt: $X \Rightarrow \bullet Y \,\|\, \bullet X \Rightarrow Y$.

Table 3. The display rules for $\delta$CL and $\delta$Kt.

$$\frac{X \Rightarrow A \qquad A \Rightarrow Y}{X \Rightarrow Y} \; (cut)$$

A *derivation* is defined in the usual way as a finite tree of display sequents constructed using the axioms and rules. Let $\delta$C be a display calculus. We write $\vdash_{\delta\mathrm{C}} P$ to mean that $\delta$C derives the formula $P$.

The display property was utilized by Belnap to prove a general cut-elimination theorem that applies whenever a display calculus satisfies the *display conditions* C1–C8. These conditions state restrictions on the inference rules of the calculus presented as rule schemata constructed from *schematic* variables for structures and formulae. A concrete instance of a rule schema is obtained by the uniform substitution of concrete structures (formulae) for corresponding schematic variables. A *parameter* is an occurrence of a schematic structure variable in a rule schema. Here we follow later presentations [68] of the display conditions which combine C6 and C7 to obtain the condition C6/7.

(C1) Each formula occurring in a premise of a rule instance is a subformula of some formula in the conclusion.

(C2) Congruent parameters are occurrences of the same structure.

(C3) Each parameter is congruent to at most one structure variable in the conclusion. That is, no two structure variables in the conclusion are congruent to each other.

(C4) Congruent parameters are all either a-part or s-part structures.

(C5) A schematic formula variable in the conclusion of an inference rule $\rho$ is either the entire antecedent or the entire succedent. This formula is called a *principal formula* of $\rho$.

(C6/7) Each inference rule is closed under simultaneous substitution of arbitrary structures for congruent parameters.

(C8) For inference rules $\rho$ and $\sigma$ with respective conclusions $X \Rightarrow A$ and $A \Rightarrow Y$ with formula $A$ principal in both inferences in the sense of C5 ('principal cut'): if *cut* is applied to yield $X \Rightarrow Y$, then

(i) $X \Rightarrow Y$ is identical to either $X \Rightarrow A$ or $A \Rightarrow Y$, or (ii) there is a derivation of $X \Rightarrow Y$ from the premises of $\rho$ and $\sigma$ such that the cut-formula of every cut is a proper subformula of $A$.

REMARK 20. *Notice that C8 is simply the reductivity condition in hypersequent calculi (cf. Definition 7) while conditions C2-C7 imply the substitutivity condition (cf. Definition 6).*

The only nontrivial display condition to verify is C8 which states that a principal cut can be reduced to cuts on smaller formulae. Conditions C5 and C8 are not relevant for structural rules since such rules do not contain schematic variables for formulae.

See the proof of Theorem 26 for a concrete example demonstrating how to check the display conditions.

EXAMPLE 21. *The following are examples of rules that violate, respectively: C3 (there are two occurrences of X in the conclusion), C4 (X is a-part and s-part; indeed so is Y) and C5 (A is not the whole of the antecedent).*

$$\frac{X \Rightarrow Y}{X \circ X \Rightarrow Y} \qquad \frac{X \Rightarrow Y}{Y \Rightarrow X} \qquad \frac{A \Rightarrow Y}{A \circ X \Rightarrow Y}$$

THEOREM 22 (Belnap). *If a display calculus satisfies C1 then it has the subformula property, that is every formula occurring in a cut-free derivation appears as a subformula of some formula in the conclusion. A display calculus satisfying C2–C8 has cut-elimination.*

PROOF. The proof proceeds Gentzen-style, successively eliminating topmost[10] instances of the cut-rule by tracing the cut-formula upwards until the cut is a principal cut. There is a subtlety here that is worth pointing out. Unlike in Gentzen's [50] proof, the multicut-rule is not required to resolve the difficulties arising from a contraction rule applied to the cut-formula. Belnap's proof avoids the use of an explicit multicut-rule in this situation by computing the set of 'ancestor' occurrences of the cut-formula and essentially applying the cut-rule to each member in that set. ■

Intuitively, the display conditions ensure that the calculus is sufficiently "well-behaved" to permit Gentzen's arguments to go through.

REMARK 23. *Although a display calculus satisfying C1 has the subformula property, note that a structure occurring in a derivation need not appear as a substructure in the conclusion, i.e., there is no substructure property.*

---

[10]A proof of strong normalization for properly displayable logics where cuts may be eliminated in any order has been formally verified in the proof-assistant Isabelle [39, 40].

### 3.1.   Classical logic and the normal tense logic Kt

We introduce the display calculus by reconstructing in steps the display calculus $\delta$Kt [68, 103] for the basic normal tense logic Kt, starting from a sequent calculus for propositional classical logic CL [100]. The logic Kt [17] extends a Hilbert calculus for CL with the modal and tense operators $\Diamond, \Box, \blacklozenge$ and $\blacksquare$, the necessitation rules $A/\Box A$, $A/\blacksquare A$, the normality axioms $\Box(A \to B) \to (\Box A \to \Box B)$, $\blacksquare(A \to B) \to (\blacksquare A \to \blacksquare B)$, converse axioms $p \to \Box\blacklozenge p$, $p \to \blacksquare\Diamond p$, and duality axioms $\Diamond A \leftrightarrow \neg\Box\neg A$ and $\blacklozenge A \leftrightarrow \neg\blacksquare\neg A$. We will motivate the addition of new structural connectives representing certain logical connectives and constants of Kt and the addition of display rules (sound for Kt) to manipulate these structural connectives in order to obtain the display property.

Let us first obtain a display calculus $\delta$CL for CL. It is well-known that in Gentzen's sequent calculus **LK** for CL, the comma in the antecedent represents conjunction and the comma in the succedent represents disjunction — indeed, a sequent $X \Rightarrow Y$ in **LK** is interpreted as the formula $\bigwedge_{A \in X} A \to \bigvee_{A \in Y} A$ (cf. Definition 3). So let us introduce[11] a new structural connective $\circ$ and logical introduction rules satisfying C5 that interpret $\circ$ as conjunction (resp. disjunction) in the antecedent (succedent).

$$\frac{A \circ B \Rightarrow X}{A \wedge B \Rightarrow X} \; (\wedge\text{l}) \qquad\qquad \frac{X \Rightarrow A \qquad Y \Rightarrow B}{X \circ Y \Rightarrow A \wedge B} \; (\wedge\text{r})$$

$$\frac{A \Rightarrow X \qquad B \Rightarrow Y}{A \vee B \Rightarrow X \circ Y} \; (\vee\text{l}) \qquad\qquad \frac{X \Rightarrow A \circ B}{X \Rightarrow A \vee B} \; (\vee\text{r})$$

In the sequent calculus **LK** just two structural connectives suffice but here we introduce another structural connective $*$, representing negation in the antecedent and the succedent.

$$\frac{*A \Rightarrow X}{\neg A \Rightarrow X} \; (\neg\text{l}) \qquad\qquad \frac{X \Rightarrow *A}{X \Rightarrow \neg A} \; (\neg\text{r})$$

Since $\to$ is definable in classical logic in terms of $\neg$ and $\vee$, we can state left and right rules for $\to$ using $*$ and $\circ$:

$$\frac{X \Rightarrow A \qquad B \Rightarrow Y}{A \to B \Rightarrow *X \circ Y} \; (\to\text{l}) \qquad\qquad \frac{X \circ A \Rightarrow B}{X \Rightarrow A \to B} \; (\to\text{r})$$

The structural constant **I** stands for $\top$ ($\bot$) in the antecedent (succedent).

$$\frac{\mathbf{I} \Rightarrow X}{\top \Rightarrow X} \; (\top\text{l}) \qquad\qquad \frac{X \Rightarrow \mathbf{I}}{X \Rightarrow \bot} \; (\bot\text{r})$$

---

[11]Instead of introducing $\circ$ we could continue to use the comma for this purpose. We use $\circ$ to distinguish the display calculus notation from that of the sequent calculus.

$X \Rightarrow Z \parallel \mathbf{I} \circ X \Rightarrow Z \qquad X \Rightarrow Z \parallel X \Rightarrow \mathbf{I} \circ Z \qquad \mathbf{I} \Rightarrow Y \parallel {*}\mathbf{I} \Rightarrow Y$

$X \Rightarrow \mathbf{I} \parallel X \Rightarrow {*}\mathbf{I} \qquad X \Rightarrow Z \,/\, Y \circ X \Rightarrow Z \qquad X \Rightarrow Z \,/\, X \circ Y \Rightarrow Z$

$X \circ Y \Rightarrow Z \,/\, Y \circ X \Rightarrow Z \quad Z \Rightarrow X \circ Y \,/\, Z \Rightarrow Y \circ X \quad X \circ X \Rightarrow Z \,/\, X \Rightarrow Z$

$Z \Rightarrow X \circ X \,/\, Z \Rightarrow X$

$X_1 \circ (X_2 \circ X_3) \Rightarrow Z \,/\, (X_1 \circ X_2) \circ X_3 \Rightarrow Z$

$Z \Rightarrow X_1 \circ (X_2 \circ X_3) \,/\, Z \Rightarrow (X_1 \circ X_2) \circ X_3$

Further structural rules for $\delta$Kt: $\mathbf{I} \Rightarrow Y \,/\, \bullet I \Rightarrow Y$ and $X \Rightarrow \mathbf{I} \,/\, X \Rightarrow \bullet\mathbf{I}$.

Table 4. Structural rules for $\delta$CL and $\delta$Kt.

Now every logical formula has a structural analogue. The calculus also contains the initial sequents $A \Rightarrow A$, $\bot \Rightarrow \mathbf{I}$, $\mathbf{I} \Rightarrow \top$ and the cut-rule.

The display rules for $\delta$CL appear in Table 3. For brevity, we use a vertical slanted line / instead of a horizontal line to separate the premises of a rule from the conclusion. The / is replaced with $\parallel$ to denote that the rule holds in both directions.

In addition to the display rules, we need further structural rules (see Table 4) to specify properties of the new structural connectives such as commutativity and associativity, and also weakening and contraction.

A formula in the language of CL is called a classical formula. A sequent in $\delta$CL has the form $X \Rightarrow Y$ where $X, Y$ belong to the following grammar.

$$X ::= A \text{ is a classical formula} \mid \mathbf{I} \mid (X \circ X) \mid {*}X$$

We interpret the sequent $X \Rightarrow Y$ as the classical formula $l(X) \rightarrow r(X)$ where the functions $l$ and $r$ map structures to classical formulae :

$$
\begin{aligned}
l(A) &= A & r(A) &= A \\
l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\
l({*}X) &= \neg r(X) & r({*}X) &= \neg l(X) \\
l(X \circ Y) &= l(X) \wedge l(Y) & r(X \circ Y) &= r(X) \vee r(Y)
\end{aligned}
$$

Let $A \in \mathcal{L}$ denote that formula $A$ is a theorem of the logic $\mathcal{L}$.

PROPOSITION 24 (Soundness). *For any classical formula $A$: if $\mathbf{I} \Rightarrow A$ is derivable in $\delta$CL then $A \in$ CL.*

PROOF. The idea is to show for every rule $\rho \in \delta\mathrm{CL}$ that if the interpretation $l(X) \to r(Y)$ of each premise $X \Rightarrow Y$ of $\rho$ is a theorem of CL then so is the interpretation of the conclusion.

For example, let $\rho$ be the display rule $X \circ Y \Rightarrow Z \,\|\, X \Rightarrow Z \circ *Y$. Recall that the double line indicates that there are actually two rules here. Let us demonstrate for the rule in the left-to-right direction. Suppose that $l(X \circ Y) \to r(Z)$, i.e., $l(X) \wedge l(Y) \to r(Z)$ is a theorem of CL. Then, arguing in classical logic it may be verified that $l(X) \to r(Z \circ *Y)$, i.e., $l(X) \to r(Z) \vee \neg r(Y)$ is a theorem of CL. ∎

PROPOSITION 25 (Completeness). *For any classical formula $A$: if $A \in \mathrm{CL}$ then $\mathbf{I} \Rightarrow A$ is derivable in $\delta\mathrm{CL}$.*

PROOF. Completeness can be shown with respect to a standard Hilbert axiomatization for CL. For each axiom schema $A$, it can be shown that $\mathbf{I} \Rightarrow A$ is derivable. Now suppose that $\vdash_{\delta\mathrm{CL}} \mathbf{I} \Rightarrow A$ and $\vdash_{\delta\mathrm{CL}} \mathbf{I} \Rightarrow A \to B$. Since $\vdash_{\delta\mathrm{CL}} A \circ (A \to B) \Rightarrow B$, using (*cut*) and the structural rules we get $\vdash_{\delta\mathrm{CL}} \mathbf{I} \Rightarrow B$. Thus $\delta\mathrm{CL}$ can simulate the rule of *modus ponens*. ∎

From the above two results it follows that $\delta\mathrm{CL}$ is a calculus for CL. Now let us obtain a calculus for Kt. It is easy to verify (for example, arguing via the Kripke semantics [17]) that

$$\blacklozenge A \to B \in \mathrm{Kt} \text{ if and only if } A \to \Box B \in \mathrm{Kt} \qquad (\mathrm{I})$$

A rule of the form $\blacklozenge A \Rightarrow B \,\|\, A \Rightarrow \Box B$, although sound, would break the subformula property (and in any case is too weak). Nevertheless, this motivates the introduction of a new structural connective $\bullet$ interpreted as $\blacklozenge$ in the antecedent and $\Box$ in the succedent to get $\blacklozenge$l and $\Box$r. Next, add rules $\blacklozenge$r and $\Box$l so that $\blacklozenge$l and $\Box$r become invertible and C8 holds.

$$\frac{\bullet A \Rightarrow X}{\blacklozenge A \Rightarrow X}\ \blacklozenge\mathrm{l} \qquad\qquad \frac{X \Rightarrow A}{\bullet X \Rightarrow \blacklozenge A}\ \blacklozenge\mathrm{r}$$

$$\frac{A \Rightarrow X}{\Box A \Rightarrow \bullet X}\ \Box\mathrm{l} \qquad\qquad \frac{X \Rightarrow \bullet A}{X \Rightarrow \Box A}\ \Box\mathrm{r}$$

Because $\Diamond A \leftrightarrow \neg\Box\neg A$ and $\blacklozenge A \leftrightarrow \neg\blacksquare\neg A$ and since $*$ represents $\neg$ we obtain:

$$\frac{* \bullet *A \Rightarrow X}{\Diamond A \Rightarrow X}\ \Diamond\mathrm{l} \qquad\qquad \frac{X \Rightarrow A}{* \bullet *X \Rightarrow \Diamond A}\ \Diamond\mathrm{r}$$

$$\frac{A \Rightarrow X}{\blacksquare A \Rightarrow * \bullet *X}\ \blacksquare\mathrm{l} \qquad\qquad \frac{X \Rightarrow * \bullet *A}{X \Rightarrow \blacksquare A}\ \blacksquare\mathrm{r}$$

See Tables 3 and 4 for the display rules and structural rules of $\delta\mathrm{Kt}$. A sequent has the form $X \Rightarrow Y$ for structures $X, Y$ from the grammar

$$X ::= A \text{ is a tense formula} \mid \mathbf{I} \mid (X \circ X) \mid *X \mid \bullet X$$

The sequent $X \Rightarrow Y$ is interpreted as the formula $l(X) \to r(Y)$ where $l$ and $r$ (page 19) are extended by $l(\bullet X) = \blacklozenge l(X)$ and $r(\bullet X) = \Box r(X)$. We need to show that the new rules are sound in Kt. As before, it suffices to show that if the interpretations of the premises are theorems of Kt, then so is the conclusion. It is helpful here to argue via the Kripke semantics for Kt. Completeness can be verified with respect to the Hilbert axiomatization.

PROPOSITION 26. *$\delta$Kt admits cut-elimination and the subformula property.*

PROOF. Cut-elimination and the subformula property follow from Theorem 22. Conditions C1–C7 can be verified on sight. Condition C8 concerns only the logical rules and corresponds to the case considered by Gentzen [50] for reducing cuts when the cut-formula is the principal formula in the last rule in each of the premise derivations.

Let us demonstrate verification of C1–C8 for rules ($\Diamond$l) and ($*$r) in $\delta$Kt:

$$\frac{* \bullet * A \Rightarrow X}{\Diamond A \Rightarrow X} \; (\Diamond \text{l}) \qquad\qquad \frac{X \circ Y \Rightarrow Z}{X \Rightarrow Z \circ * Y} \; (*\text{r})$$

Consider arbitrary instances of these rules. It is easy to see that any formula occurring in the premise must occur as a subformula of some formula in the conclusion. Thus C1 is satisfied. As for C2, notice that the symbol $X$ occurs in the premise and conclusion of ($\Diamond$l). These occurrences are *congruent*. Similarly, the occurrences of $X$ (also $Y, Z$) in the premise and conclusion of ($*$r) are congruent. Each distinct schematic structure variable in ($\Diamond$l) and ($*$r) appears exactly once in the conclusion, so C3 holds. C4 can be verified by using the display rules to display every occurrence of a schematic structure variable. In the conclusion of ($\Diamond$l): $\Diamond A$ is the whole of the antecedent, so C5 holds. Since ($*$r) is a structural rule, conditions C5 and C8 are not relevant for this rule. There are no side-conditions restricting instantiation of structures in the rules, so C6/C7 holds immediately. C8: Observe that ($\Diamond$l) makes $\Diamond A$ principal in the antecedent. Now the only rule in $\delta$Kt (see the rules following equation (I) on page 20) that can make $\Diamond A$ principal in the succedent is ($\Diamond$r). This is the situation below left. To show C8 it suffices to derive the same sequent using (*cut*) on proper subformulae of $\Diamond A$ (below right). Here drs denotes some number of display rules.

$$\frac{\dfrac{X \Rightarrow A}{* \bullet * X \Rightarrow \Diamond A} \Diamond \text{r} \quad \dfrac{* \bullet * A \Rightarrow Y}{\Diamond A \Rightarrow Y} \Diamond \text{l}}{* \bullet * X \Rightarrow Y} \qquad \frac{X \Rightarrow A \quad \dfrac{\dfrac{* \bullet * A \Rightarrow Y}{A \Rightarrow * \bullet * Y} \text{drs}}{X \Rightarrow * \bullet * Y}}{* \bullet * X \Rightarrow Y} \text{drs}$$

$\blacksquare$

Since the tense logic Kt is conservative over the basic normal modal logic K, from cut-elimination it is easy to see that the calculus obtained from $\delta$Kt by deleting the introduction rules for $\blacklozenge$ and $\blacksquare$ is a display calculus $\delta$K for K. Notice that in general, a sequent appearing in a derivation from $\delta$K will be interpreted as a *tense* (not necessarily modal) formula.

REMARK 27. *If $\mathcal{C}$ is a display calculus for a logic in the language $\mathcal{L}$, calculi for fragments $\mathcal{L}'$ of $\mathcal{L}$ can be obtained by deleting the rules introducing the other logical connectives. If the original calculus satisfies C1–C8, then any subcalculus $\mathcal{C}'$ will also satisfy C1–C8. Observe that in general, a sequent in $\mathcal{C}'$ will be interpreted as a formula from the larger language $\mathcal{L}$.*

### 3.2. How to use residuation to construct a display calculus

In the previous section we motivated the construction of a display calculus $\delta$Kt for Kt starting from a sequent calculus for CL. The intention was to give a "hands-on" introduction to the display calculus and its main features (display property, display conditions). This construction gives rise to various questions. How many new structural connectives do we need and which logical connectives should these represent? How are we to choose the display rules? Could we directly construct a display calculus for the modal logic K instead of taking a detour via the tense logic Kt?

To address these questions, recall first how we added the modal and tense operators to get $\delta$Kt from $\delta$CL. We began by identifying the property (I). This immediately suggested the introduction of a new structural connective $\bullet$ to stand for $\blacklozenge$ in the antecedent and $\square$ in the succedent. The rules $\blacklozenge$l and $\square$r were immediate. Although we did not discuss it there, the rule for $\blacklozenge$r (the case of $\square$l is similar) could be obtained by considering what rule is needed to satisfy condition C8 and to make $\blacklozenge$l invertible: given $\blacklozenge A \Rightarrow X$ we want to derive $\bullet A \Rightarrow X$. Arguing backwards from the conclusion, consider the introduction of a cut (below left). How to derive $\bullet A \Rightarrow \blacklozenge A$? Since $A \Rightarrow A$ is derivable, we are led to the the rule $\blacklozenge$r (below center). Now it may be seen (below right) that $\blacklozenge$l is indeed invertible.

$$\frac{\bullet A \Rightarrow \blacklozenge A \quad \blacklozenge A \Rightarrow X}{\bullet A \Rightarrow X} \ (cut) \qquad \frac{X \Rightarrow A}{\bullet X \Rightarrow \blacklozenge A} \ \blacklozenge\text{r} \qquad \frac{\dfrac{A \Rightarrow A}{\bullet A \Rightarrow \blacklozenge A} \ \blacklozenge\text{r} \quad \blacklozenge A \Rightarrow X}{\bullet A \Rightarrow X} \ (cut)$$

The property (I) also led to the display rules $X \Rightarrow \bullet Y \parallel \bullet X \Rightarrow Y$. Finally we added further structural rules to capture the intended behavior of $\bullet$ (see Table 4). A statement of the form (I) is an example of a *residuation property*, see, e.g., [52]. It should now be clear why we introduced a calculus

for Kt and not K: in order to represent the residuation property (I) we need both $\square$ *and* $\blacklozenge$. We say that $\bullet$ is a *Gentzen toggle* [68] for the residuated pair $(\blacklozenge, \square)$.[12]

It has been emphasized in [52,88,105] that residuation is the central idea behind constructing display calculi. The residuation property is the key to identifying new structural connectives and display rules. We may start with the initial sequent $A \Rightarrow A$ and the cut-rule. Then:

1. Identify the residuation properties for the logic of interest. Let us suppose that for binary logical connectives $\lhd, \rhd$ the residuation property has the following form:

$$A \Rightarrow B \rhd C \text{ iff } A \lhd B \Rightarrow C \text{ iff } B \Rightarrow A \rhd C$$

   The residuated pair is $(\lhd, \rhd)$.

2. Add a new structural connective $\star$ to represent the connective $\lhd$ in the antecedent and $\rhd$ in the succedent.

3. This leads to the following logical 'rewrite' [52] rules.

$$\frac{A \star B \Rightarrow X}{A \lhd B \Rightarrow X} \lhd l \qquad\qquad \frac{X \Rightarrow A \star B}{X \Rightarrow A \rhd B} \rhd r$$

4. We are missing the logical rules $\lhd r$ and $\rhd l$. Choose the form of these rules so that $\lhd l$ and $\rhd r$ become invertible and C8 is satisfied; see [52] for the technical procedure. Typically such 'decoding' rules are not invertible.

5. Use the residuation property to read off the display rules. For the residuation property under consideration we get the following display rules:

$$\frac{\dfrac{X \Rightarrow Y \star Z}{X \star Y \Rightarrow Z}}{Y \Rightarrow X \star Z}$$

6. Finally, introduce (where necessary and possible) further structural rules manipulating $\star$ to capture the logic of interest. A method for introducing structural rules from suitable axioms is given in Section 4.2.2.

These ideas were introduced in [52] where it is shown how to construct a display calculus $\delta L$ satisfying C1–C8 from a logic $L$ that is defined by residuation and satisfying certain other properties. The construction was used to systematically construct display calculi for various substructural logics [53]. Indeed, this approach can be applied to rich vocabularies, including, for example, the language of dynamic epistemic logic, see [47].

---

[12]See, e.g. [49] for an algebraic account of residuated pairs in the context of the sequent calculus.

### 3.3.   Case study: Bi-intuitionistic logic

Let us demonstrate how to construct a display calculus $\delta$Bi-IL [109] for Bi-intuitionistic logic Bi-IL following the above approach. The logic Bi-IL is an extension of propositional intuitionistic logic IL with the dual $\leftarrow_d$ connective of $\rightarrow$ and Rauszer's axioms [87], see also [111]. It is an extension of non-associative non-commutative Bi-Lambek logic Bi-FL with the axioms for exchange, contraction, weakening and associativity. Strictly speaking, to follow the approach above we should first obtain a display calculus for Bi-FL and then add structural rules corresponding to the axioms. However, to simplify matters[13] we will state the residuation properties as they hold in Bi-IL:

$$A \Rightarrow (B \rightarrow C) \quad \text{iff} \quad A \wedge B \Rightarrow C \quad \text{iff} \quad B \Rightarrow (A \rightarrow C)$$
$$(C \leftarrow_d A) \Rightarrow B \quad \text{iff} \quad C \Rightarrow A \vee B \quad \text{iff} \quad (C \leftarrow_d B) \Rightarrow A$$

Assign the structural connective $\circ$ for $(\wedge, \rightarrow)$ and $\bullet$ for $(\leftarrow_d, \vee)$. Then we have the following rewrite rules:

$$\frac{A \circ B \Rightarrow Y}{A \wedge B \Rightarrow Y} \wedge \mathrm{l} \qquad\qquad \frac{X \Rightarrow A \circ B}{X \Rightarrow A \rightarrow B} \rightarrow\mathrm{r}$$

$$\frac{B \bullet A \Rightarrow Y}{B \leftarrow_d A \Rightarrow Y} \leftarrow_d\mathrm{l} \qquad\qquad \frac{X \Rightarrow A \bullet B}{X \Rightarrow A \vee B} \vee\mathrm{r}$$

The following decoding rules yield invertibility of the rewrite rules and compliance with condition C8.

$$\frac{X \Rightarrow A \qquad Y \Rightarrow B}{X \circ Y \Rightarrow A \wedge B} \qquad\qquad \frac{X \Rightarrow A \qquad B \Rightarrow Y}{A \rightarrow B \Rightarrow X \circ Y} \rightarrow\mathrm{l}$$

$$\frac{X \Rightarrow B \qquad A \Rightarrow Y}{X \bullet Y \Rightarrow B \leftarrow_d A} \leftarrow_d\mathrm{r} \qquad\qquad \frac{A \Rightarrow X \qquad B \Rightarrow Y}{A \vee B \Rightarrow X \bullet Y}$$

LEMMA 28.  *The rules $\wedge l$, $\rightarrow r$, $\leftarrow_d l$ and $\vee r$ are invertible.*

Using the residuation properties we obtain the display rules:

$$\frac{\dfrac{X \Rightarrow Y \circ Z}{X \circ Y \Rightarrow Z}}{Y \Rightarrow X \circ Z} \qquad\qquad \frac{\dfrac{X \bullet Y \Rightarrow Z}{X \Rightarrow Y \bullet Z}}{X \bullet Z \Rightarrow Y}$$

---

[13]The language of Bi-FL contains the following logical connectives and constants: $\wedge, \top, \otimes, \mathbf{1}, \vee, \bot, \oplus, \mathbf{0}, \rightarrow, \leftarrow, \rightarrow_d, \leftarrow_d$. The residuation properties take the following form:

$$A \Rightarrow (C \leftarrow B) \quad \text{iff} \quad A \otimes B \Rightarrow C \quad \text{iff} \quad B \Rightarrow (A \rightarrow C)$$
$$(A \rightarrow_d C) \Rightarrow B \quad \text{iff} \quad C \Rightarrow A \oplus B \quad \text{iff} \quad (C \leftarrow_d B) \Rightarrow A$$

In Bi-IL the following pairs conflate: $\wedge, \otimes$; $\vee, \oplus$; $\rightarrow, \leftarrow$; $\rightarrow_d, \leftarrow_d$; $\top, \mathbf{1}$; $\bot, \mathbf{0}$.

We add rewrite rules for the structural constant $\mathbf{I}$ and their decoding rules:

$$\frac{\mathbf{I} \Rightarrow X}{\top \Rightarrow X} \top\mathrm{l} \qquad \frac{X \Rightarrow \mathbf{I}}{X \Rightarrow \bot} \bot\mathrm{r} \qquad \frac{}{\mathbf{I} \Rightarrow \top} \top\mathrm{r} \qquad \frac{}{\bot \Rightarrow \mathbf{I}} \bot\mathrm{l}$$

The constants $\top$ and $\bot$ are characterized by the axioms $\top \wedge A \leftrightarrow A \leftrightarrow A \wedge \top$ and $\bot \vee A \leftrightarrow A \leftrightarrow A \vee \bot$ respectively. The corresponding structural rules are the following (these can be obtained using the method of Section 4.2.2):

$$\frac{\dfrac{\mathbf{I} \circ X \Rightarrow Y}{X \Rightarrow Y}}{X \circ \mathbf{I} \Rightarrow Y} \qquad\qquad \frac{\dfrac{X \Rightarrow Y \bullet \mathbf{I}}{X \Rightarrow Y}}{X \Rightarrow \mathbf{I} \bullet Y}$$

It remains to add the structural rules for exchange, associativity, weakening and contraction. It is easy to see that the following suffice

$$\frac{X \Rightarrow Y}{X \Rightarrow Y \bullet Z} \qquad \frac{X \Rightarrow Y}{X \circ Z \Rightarrow Y} \qquad \frac{X \Rightarrow Y \bullet Z}{X \Rightarrow Z \bullet Y} \qquad \frac{X \circ Z \Rightarrow Y}{Z \circ X \Rightarrow Y}$$

$$\frac{X \Rightarrow Y \bullet Y}{X \Rightarrow Y} \qquad \frac{X \circ X \Rightarrow Y}{X \Rightarrow Y} \qquad \frac{X \Rightarrow (Y \bullet Z) \bullet U}{X \Rightarrow Y \bullet (Z \bullet U)} \qquad \frac{(X \circ Y) \circ Z \Rightarrow U}{X \circ (Y \circ Z) \Rightarrow U}$$

REMARK 29. *In this particular case, we can make the rules introducing $\wedge$ (resp. $\vee$) in the succedent (antecedent) invertible by adapting them slightly:*

$$\frac{X \Rightarrow A \qquad X \Rightarrow B}{X \Rightarrow A \wedge B} \wedge r \qquad\qquad \frac{A \Rightarrow X \qquad B \Rightarrow X}{A \vee B \Rightarrow X} \vee l$$

It is easy to verify that $\delta$Bi-IL satisfies C1–C8. The construction immediately suggests interpreting a sequent $X \Rightarrow Y$ as the formula $l(X) \to r(Y)$.

$$
\begin{aligned}
l(A) &= A & r(A) &= A \\
l(\mathbf{I}) &= \top & r(\mathbf{I}) &= \bot \\
l(X \circ Y) &= l(X) \wedge l(Y) & r(X \circ Y) &= l(X) \to r(Y) \\
l(X \bullet Y) &= l(X) \leftarrow_d r(Y) & r(X \bullet Y) &= r(X) \vee r(Y)
\end{aligned}
$$

Since the logic Bi-IL is conservative over IL, deleting the rules $\leftarrow_d l$ and $\leftarrow_d r$ from $\delta$Bi-IL yields a calculus $\delta$IL for IL satisfying C1–C8.

## 3.4. First-order quantifiers

It appears that [105, Chapter 12], [107] is the only investigation into first-order display calculi that goes beyond certain comments by Belnap [15, p. 408], who remarks that "the obvious rules:

$$(UQ) \qquad \frac{Aa \vdash X}{(x)Ax \vdash X} \qquad\qquad \frac{X \vdash Aa}{X \vdash (x)Ax}$$

provided, for the right rule, that $a$ does not occur free in the conclusion" together with the dual rules for the existential quantifier provide "no extra illumination". According to Belnap,

> that is because these rules for quantifiers are "structure free" (no structure connectives are involved; ...). One upshot is that adding these quantifiers to modal logic brings along Barcan and its converse ... willy-nilly, which is an indication of an unrefined account; alternatives need investigating.

Structure-involving left and right introduction rules for the quantifiers are obtained in [105, 107] by regarding predicate logic as a kind of propositional modal logic, see [70, 101]. The key observation is that $(\exists x, \forall x)$, like $(\blacklozenge, \square)$, forms a residuated pair of operations. Adjointness between $\exists x$ and $\forall x$ has been highlighted by Goldblatt [51, Chapter 15]. Let $R_x$ be a binary relation on some non-empty set of states $S$. Define functions on the powerset of $S$:

$$
\begin{array}{rcl}
\forall x\mathsf{A} & := & \{a \mid \forall b\ (aR_xb \text{ implies } b \in \mathsf{A})\} \\
\exists x\mathsf{A} & := & \{a \mid \exists b\ (aR_xb \text{ and } b \in \mathsf{A})\} \\
\forall x^{\smile}\mathsf{A} & := & \{a \mid \forall b\ (bR_xa \text{ implies } b \in \mathsf{A})\} \\
\exists x^{\smile}\mathsf{A} & := & \{a \mid \exists b\ (bR_xa \text{ and } b \in \mathsf{A})\}
\end{array}
$$

These functions satisfy:

$$
\exists x^{\smile}\mathsf{A} \subseteq \mathsf{B} \text{ iff } \mathsf{A} \subseteq \forall x\mathsf{B}, \qquad \exists x\mathsf{A} \subseteq \mathsf{B} \text{ iff } \mathsf{A} \subseteq \forall x^{\smile}\mathsf{B}.
$$

Van Benthem [101] suggests viewing variable assignments $\alpha, \beta, \ldots$ in first-order models $\mathcal{M}$ as elements of a non-empty set of states $S$. The binary relation $=_x$ that holds between two variable assignments $\alpha$ and $\beta$ if and only if they are $x$-variants of each other, i.e., if and only if they are identical up to possibly assigning different individuals to variable $x$, allows one to write Tarski's satisfaction clauses for universally and existentially quantified formulae in the style of the truth clauses for $\lozenge, \square$ in Kripke models:

$$
\begin{array}{lll}
\mathcal{M}, \alpha \models \exists x A & \text{iff} & \text{there is a } \beta \in S \text{ with } \alpha =_x \beta \text{ and } \mathcal{M}, \beta \models A \\
\mathcal{M}, \alpha \models \forall x A & \text{iff} & \text{for every } \beta \in S: \text{ if } \alpha =_x \beta, \text{ then } \mathcal{M}, \beta \models A
\end{array}
$$

Since $=_x$ is symmetric, there is no distinction between backward-looking and forward-looking quantifier prefixes, as in the general case of relations $R_x$. The general case gives rise to a decidable minimal first-order modal logic $\mathrm{K}^{fo}$. Display calculi for extensions of $\mathrm{K}^{fo}$ are considered in [105, 107].

We consider a first-order language without function symbols and individual constants. The idea now is to introduce for every individual variable $x$ a

structure connective $\bullet_x$. These denumerably many structure connective are assumed to satisfy the following display equivalence: $X \Rightarrow \bullet_x Y \parallel \bullet_x X \Rightarrow Y$. The left and right introduction rules for $\forall x$ and $\exists x$ are:

$$\frac{A \Rightarrow X}{\forall x A \Rightarrow \bullet_x X} \ \forall x \mathrm{l} \qquad \frac{X \Rightarrow \bullet_x A}{X \Rightarrow \forall x A} \ \forall x \mathrm{r}$$

$$\frac{* \bullet_x * A \Rightarrow X}{\exists x A \Rightarrow X} \ \exists x \mathrm{l} \qquad \frac{X \rightarrow A}{* \bullet_x * X \Rightarrow \exists_x A} \ \exists x \mathrm{r}$$

So far, the treatment of the quantifiers takes into account only the adjointness between $\exists x$ and $\forall x$ but in no way their variable-binding. We therefore extend the operation $[y/z]A$ of substituting $y$ for every free occurrence of $z$ in $A$ to arbitrary structures by stipulating $[y/z]\mathbf{I} = \mathbf{I}$ and letting $[y/z]$ commute with $*$, $\circ$ and $\bullet_x$, for every variable $x$.

As additional structural rules we assume the structural counterparts of necessitation, $\mathbf{I} \Rightarrow Y / \bullet_x \mathbf{I} \Rightarrow Y$, $X \Rightarrow \mathbf{I} / X \Rightarrow \bullet_x \mathbf{I}$ together with the following rules:

$$\begin{array}{lll} \mathrm{r}1.3 & X \Rightarrow Y / X \Rightarrow \bullet_x Y, & \text{if } x \text{ does not occur free} \\ & & \text{in any formula in } Y \\ \mathrm{r}1.4 & X \Rightarrow \bullet_x Y / X \Rightarrow [y/x]Y, & \text{if } y \text{ is free for } x \text{ in every} \\ & & \text{formula in } Y \end{array}$$

Let $\delta\mathrm{CL}^{fo}$ be the result of adding the above sequent rules to $\delta\mathrm{CL}$, including (*cut*) and initial sequents $A \Rightarrow A$, for atomic formulae $A$. It follows by induction on the structure of $A$ that $A \Rightarrow A$ is provable in $\delta\mathrm{CL}^{fo}$ for arbitrary $A$. Moreover, it can be shown that $\delta\mathrm{CL}^{fo}$ is sound and complete with respect to classical first-order logic $\mathrm{CL}^{fo}$.

PROPOSITION 30 (Soundness). *For any classical first-order formula $A$: if $\mathbf{I} \rightarrow A$ is derivable in $\delta\mathrm{CL}^{fo}$ then $A \in \mathrm{CL}^{fo}$.*

PROOF. We extend the translation of display sequents into classical formulae by setting $l(\bullet_x X) = \exists x l(X)$ and $r(\bullet_x X) = \forall x r(X)$ and show that the rules of $\delta\mathrm{CL}^{fo}$ that are not already rules of $\delta\mathrm{CL}$ are validity-preserving under this extended translation. Display equivalences: Assume that (i) the universal closure of $l(X) \rightarrow r(\bullet_x Y)$ is valid, but the universal closure of $l(\bullet_x X) \rightarrow r(Y)$ is not. Then there is a first-order model $\mathcal{M}$ and an assignment $\alpha$ in $\mathcal{M}$ with $\mathcal{M}, \alpha \models \exists x l(X)$ and (ii) $\mathcal{M}, \alpha \not\models r(Y)$. Therefore, there is an assignment $\beta$ of which $\alpha$ is an $x$-variant with $\mathcal{M}, \beta \models l(X)$. By (i), for every $x$-variant $\gamma$ of $\beta$, $\mathcal{M}, \gamma \models r(Y)$, in contradiction with (ii). Conversely, assume that

(i) the universal closure of $l(\bullet_x X) \to r(Y)$ is valid, but the universal closure of $l(X) \to r(\bullet_x Y)$ is not. Then there is a first-order model $\mathcal{M}$ and an assignment $\alpha$ in $\mathcal{M}$ with $\mathcal{M}, \alpha \not\models \forall x r(Y)$ and (ii) $\mathcal{M}, \alpha \models l(X)$. Therefore, there is an $x$-variant $\beta$ of $\alpha$ with $\mathcal{M}, \beta \not\models r(Y)$. By (i), $\mathcal{M}, \beta \models \neg\exists x l(X)$. Thus, for every assignment $\gamma$ in $\mathcal{M}$ of which $\beta$ is an $x$-variant, $\mathcal{M}, \gamma \not\models l(X)$, and we obtain a contradiction with (ii). The right and left rules for $\forall x$ and $\exists x$ and the rules r1.3 and r1.4 are either obvious or require only simple calculation (as in the case of the display equivalences). ∎

PROPOSITION 31 (Completeness). *For any classical first-order formula $A$: if $A \in \mathrm{CL}^{fo}$, then $\mathbf{I} \to A$ is derivable in $\delta\mathrm{CL}^{fo}$.*

PROOF. We consider the axiomatization of $\mathrm{CL}^{fo}$ in [44], which is particularly apt from our modal perspective:

  1.1   all universal closures of instances of tautology schemata and of
        instances of the following schemata 1.2–1.4.
  1.2   $\forall x(A \to B) \to (\forall x A \to \forall x B)$
  1.3   $A \to \forall x A$, if no occurrence of $x$ is free in $A$
  1.4   $\forall x A \to [y/x]A$, if $y$ is free for $x$ in $A$
  1.5   *modus ponens*

*Modus ponens* is dealt with as in the case of $\delta\mathrm{CL}$. To derive the axioms according to 1.1, the structural necessitation rule $\mathbf{I} \Rightarrow Y \;/\; \bullet_x \mathbf{I} \Rightarrow Y$ may be used. The distribution axiom 1.2 is derived as follows; we highlight some applications of structural rules:

$$
\cfrac{
  \cfrac{
    \cfrac{
      \cfrac{
        \cfrac{
          \cfrac{
            \cfrac{
              \cfrac{
                \cfrac{
                  \cfrac{
                    \cfrac{
                      \cfrac{
                        \cfrac{A \Rightarrow A}{\forall x A \Rightarrow \bullet_x A}
                      }{\forall x(A \to B) \circ \forall x A \Rightarrow \bullet_x A} \text{ weakening} \qquad B \Rightarrow B
                    }{\bullet_x(\forall x(A \to B) \circ \forall x A) \Rightarrow A}
                  }{A \to B \Rightarrow * \bullet_x (\forall x(A \to B) \circ \forall x A) \circ B}
                }{\forall x(A \to B) \Rightarrow \bullet_x(* \bullet_x (\forall x(A \to B) \circ \forall x A) \circ B)}
              }{\forall x(A \to B) \circ \forall x A \Rightarrow \bullet_x(* \bullet_x (\forall x(A \to B) \circ \forall x A) \circ B)} \text{ weakening}
            }{\bullet_x(\forall x(A \to B) \circ \forall x A) \Rightarrow * \bullet_x (\forall x(A \to B) \circ \forall x A) \circ B}
          }{\bullet_x(\forall x(A \to B) \circ \forall x A) \circ \bullet_x(\forall x(A \to B) \circ \forall x A) \Rightarrow B}
        }{\bullet_x(\forall x(A \to B) \circ \forall x A) \Rightarrow B} \text{ contraction}
      }{\forall x(A \to B) \circ \forall x A \Rightarrow \bullet_x B}
    }{\forall x(A \to B) \circ \forall x A \Rightarrow \forall x B}
  }{\forall x(A \to B) \Rightarrow (\forall x A \to \forall x B)}
}{\mathbf{I} \circ \forall x(A \to B) \Rightarrow (\forall x A \to \forall x B)}
$$

$$
\mathbf{I} \Rightarrow \forall x(A \to B) \to (\forall x A \to \forall x B)
$$

The axiom schemata 1.3 and 1.4 can be proved straightforwardly using the rules r1.3 and r1.4. ∎

Let $\delta\mathrm{K}^{fo}$ be the result of dropping r1.3 and r1.4 from $\delta\mathrm{CL}^{fo}$. Cut-elimination for $\delta\mathrm{K}^{fo}$ follows from Belnap's general cut-elimination result. The rules r1.3 and r1.4, however, contain side-conditions that result in a violation of constraint (C6/7).

EXAMPLE 32. *To give another example of a derivation in $\delta\mathrm{CL}^{fo}$, we prove the constant domain schema $\forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$, where $x$ does not occur free in $B$. We may apply the rule r1.3 to prove $\forall x(A(x) \vee B) \vee \neg B \Rightarrow \forall x(\forall x(A(x) \vee B) \vee \neg B)$:*

$$\frac{\dfrac{\forall x(A(x) \vee B) \vee \neg B \Rightarrow \forall x(A(x) \vee B) \vee \neg B}{\forall x(A(x) \vee B) \vee \neg B \Rightarrow \bullet_x(\forall x(A(x) \vee B) \vee \neg B)}}{\forall x(A(x) \vee B) \vee \neg B \Rightarrow \forall x(\forall x(A(x) \vee B) \vee \neg B)} \ \text{r1.3}$$

*Let us refer to this derivation as $\mathcal{D}$. Moreover, we need rule r1.4:*

$$
\frac{
\frac{
\frac{
\frac{
\frac{\dfrac{A(x) \Rightarrow A(x) \qquad B \Rightarrow B}{A(x) \vee B \Rightarrow A(x) \circ B}}{\forall x(A(x) \vee B) \Rightarrow \bullet_x(A(x) \circ B)}
}{\forall x(A(x) \vee B) \Rightarrow A(x) \circ B} \ \text{r1.4}
}{\forall x(A(x) \vee B) \circ *B \Rightarrow A(x)}
}{\raise2pt\hbox{$\vdots$}}
}{
\frac{
\frac{
\mathcal{D} \quad \dfrac{\dfrac{\forall x(A(x) \vee B) \wedge \neg B \Rightarrow A(x)}{\forall x(\forall x(A(x) \vee B) \wedge \neg B) \Rightarrow \bullet_x A(x)}}{\forall x(\forall x(A(x) \vee B) \wedge \neg B) \Rightarrow \forall x A(x)}
}{\forall x(A(x) \vee B) \wedge \neg B \Rightarrow \forall x A(x)} \ \text{cut}
}{
\frac{\frac{\frac{\frac{\frac{\raise2pt\hbox{$\vdots$}}{\forall x(A(x) \vee B) \circ *B \Rightarrow \forall x A(x)}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \circ B}}{\forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}}{\mathbf{I} \circ \forall x(A(x) \vee B) \Rightarrow \forall x A(x) \vee B}}{\mathbf{I} \Rightarrow \forall x(A(x) \vee B) \to (\forall x A(x) \vee B)}
}
}
$$

Note that the earlier mentioned Barcan formula and its converse,

$$(BF) \ \forall x \Box A \to \Box \forall x A, \qquad (BFc) \ \Box \forall x A \to \forall x \Box A$$

cannot be proved in the modal extension of $\delta\mathrm{K}^{fo}$ obtained by adding the rules $\Box\mathrm{r}$, $\Box\mathrm{l}$ and the display and structural rules for $\bullet$. These schematic

formulae correspond to the structural rules

$$\mathrm{r}BF \ \ X \Rightarrow \bullet_x \bullet Y \ / \ X \Rightarrow \bullet \bullet_x Y, \qquad \mathrm{r}BFc \ \ X \Rightarrow \bullet \bullet_x Y \ / \ X \Rightarrow \bullet_x \bullet Y$$

respectively, in the sense that $BF$ ($BFc$) is provable in the presence of $\mathrm{r}BF$ ($\mathrm{r}BFc$) and $\mathrm{r}BF$ ($\mathrm{r}BFc$) is validity preserving on a first-order Kripke frame under the translation of display sequents if the frame satisfies anti-monotonicity (monotonicity) of domains, the frame condition corresponding with $BF$ ($BFc$).

## 4.   Comparing hypersequent and display calculi

We have seen that the hypersequent and display calculus can be viewed as extensions of the sequent calculus obtained via the addition of structural connectives. In the case of hypersequents, the generalisation is minimal; instead of using a single sequent, multiple sequents are used, separated by the structural connective $|$. For this reason, the simplicity of the sequent calculus largely carries over to the hypersequent calculus, resulting in a range of applications.

Nevertheless, powerful as it is, the hypersequent calculus cannot capture all interesting logics in a cut-free manner. Sometimes ad hoc generalizations of hypersequents are enough, cf. [99] for Corsi's logic (see Section 2.3) or the machinery in [82] for modal logics (tree hypersequents, see Section 5), while often this does not seem to be the case, e.g., for bi-intuitionistic logic (Section 3.3) or for the family of Bunched Logics [18]. The display calculus extends the sequent calculus in a different direction, and much further, by adding a number of structural connectives – instead of just one – and new rules in order to obtain the display property. This feature supports the development of a proof-theoretic semantics of the logical operations (Section 4.3) and makes it possible to give analytic calculi for larger[14] classes of logics.

Notice that in contrast to hypersequents where the structural connective $|$ is used ('externally') to separate the sequents, in the display calculus new structural connectives are added ('internally') to the formulae. The latter leads to greater expressiveness but the price to pay is that there is no *real* subformula property, as expressed in [4]. This renders the extraction

---

[14]Translations of hypersequents into display sequents were first considered in [105, Chapter 11], [106]. Recent results [86] show that the hypersequent calculus can be embedded in the display calculus. I.e., the latter framework is expressive enough to allow the construction of a display calculus from any concrete hypersequent calculus. The construction preserves proof-theoretic properties such as cut-elimination and the subformula property.

of information from cut-free display calculi more difficult than in standard sequent or hypersequent calculi.

In this section we illustrate these facts by reviewing some applications of the two frameworks (Section 4.1) and describing recent methods to define modular cut-free hypersequent and display calculi for large classes of logics. Although these methods are essentially the same, we capture more axiomatic extensions via structural rule extension starting from a display calculus.

### 4.1. Some applications

Cut-free hypersequent and display calculi provide a suitable framework for presenting a wide range of logics. These frameworks permit better proof search than Hilbert systems and make it easier to prove various properties of these logics. For instance, consistency is often an easy corollary of the redundancy of the cut-rule. We recall below a number of applications of cut-free hypersequent (HC) and display calculi (DC).

(HC) **Admissible rules:** The admissible rules of a logic (understood as a structural consequence relation) may be described as rules that can be added to the logic without producing any new theorems. Cut-free hypersequent calculi have been used to prove the admissibility of various rules in a number of logics. For instance, the admissibility of the disjunctive syllogism (if $\vdash \neg A$ and $\vdash A \vee B$ then $\vdash B$) has been considered as one of the major problems in the family of logics known as relevance logics. The first *constructive* proof for the relevant logic RM is a simple corollary of the cut-elimination theorem for its hypersequent calculus, see [2, 4].

Furthermore, the hypersequent calculi for various modal and intermediate logics have been used in [63, 64] to define proof systems for deriving all the admissible rules of the formalized logics.

(HC) **Standard Completeness**: A logic is standard complete when it is complete with respect to algebras based on truth values in the real unit interval $[0, 1]$. Standard complete logics have been receiving increasing attention in the last years as the formal counterpart of Fuzzy Logic, see [62]. Given a logic $\mathcal{L}$ described as a Hilbert system, discovering whether $\mathcal{L}$ is standard complete can be a challenging task using a model theoretic approach (see e.g. [35]), and the proof is inherently logic-specific. On the other hand if $\mathcal{L}$ possesses a suitable cut-free calculus, say $C\mathcal{L}$, the main step in the proof of standard completeness is to show that a special rule called density[15] is admissible in $C\mathcal{L}$. In hypersequent calculi (an instance of) this

---

[15]The density rule was introduced by Takeuti and Titani in their axiomatization of first-

rule has the form:

$$\frac{G' \mid \Sigma, p \Rightarrow \Pi \mid \Lambda \Rightarrow p}{G' \mid \Sigma, \Lambda \Rightarrow \Pi} \; (D)$$

(where $p$ is a propositional variable not occurring in $\Sigma, \Lambda, \Pi$ or $G'$) and the proof of its redundancy (in fact, of its elimination) proceeds in a similar manner to a proof of cut-elimination [13, 30, 72]. In contrast with the algebraic proofs of standard completeness, the proofs using hypersequents apply to large classes of logics provided that their calculi satisfy suitable syntactic conditions [13, 30] that can be easily verified on sight or checked using the program *AxiomCalc*, see Section 4.2.1. Furthermore, for logics whose algebraic models are not integral (or equivalently, in which the usual weakening rules are not sound) this is the only known approach, see [30, 72].

(HC) **Herbrand theorem:** The hypersequent calculi for Gödel logic and for its contraction-free counterpart (i.e., monoidal t-norm based logic MTL [45]) have been used in [7, 12] to prove Herbrand's theorem for their prenex fragment. In both cases the key result is the proof that in these calculi a certain separation between propositional and quantifier inferences can be achieved in deriving a prenex hypersequent (*mid-hypersequent theorem*). Note that the analogous result does not hold for Gentzen's **LJ** [96] and for its contraction-free counterpart, and it is achieved in the hypersequent calculi of Gödel logic and MTL using the communication rule (*com*) (see Table 2).

(DC) **Interpolation:** Unlike in the case of the sequent calculus, due to the presence of the (ec) rule, cut-elimination for a hypersequent calculus does not usually imply the Craig interpolation theorem [4]. For example, it is not known whether first-order Gödel logic interpolates, despite of its cut-free calculus. Display calculi have instead been used in [19] to give alternative proofs of the Craig interpolation theorem for multiplicative linear logic, multiplicative additive linear logic and classical logic. Though there exist sequent calculi with interpolation for these logics, [19] shows that when well-designed, display calculi *can* be used to prove interpolation.

(DC) **Decidability:** In general, the presence of many structural connectives and display equivalences detracts from the usefulness of cut-elimination in display calculi, because the subformula property is not accompanied by a substructure property. Nevertheless, some *new* decidability results have

---

order Gödel logic [97]. Ignoring $\Sigma$, in semantic terms $(D)$ can be read contrapositively as saying (very roughly) that there exists an assignment of truth values such that for some propositional variable $p$ the value of $p$ is strictly between the values of $\Lambda$ and $\Pi$; hence the name "density".

been obtained using display calculi. For instance, Restall [89] proved decidability of various properly displayable extensions of the weak relevant logic **DW** by defining a notion of semi-reduced display sequents and irredundant semi-reduced proofs. In [104], [105, Chapter 6] display calculus presentations are used to show decidability of the modal logic of functional accessibility relations and deterministic propositional dynamic logic without the Kleene-star. More recently, decidability and complexity of Full Intuitionistic Linear Logic have been settled [36] starting from a display calculus for this logic, thus solving a problem open since 1994. Note that Kracht [68] proved that it is undecidable whether or not a properly displayable logic is decidable.

## 4.2. Structural rule extensions

A common feature of hypersequent and display calculi is the existence of algorithms to transform large classes of Hilbert axioms into equivalent *structural rules* that preserve cut-elimination when added to a suitable base calculus [26,33,34,68]. This permits the automated introduction of hypersequent and display calculi for large classes of logics in a modular and systematic way. Since the rules to be added are devoid of any logical connectives, this fulfils a desideratum in [4] for good proof systems. The results [26, 34, 68] start with a *specific logic* and introduce cut-free calculi for (some of) its axiomatic extensions, e.g., intuitionistic logic without weakening and contraction for the hypersequent calculi in [26], its involutive counterpart [34], and the tense logic Kt for the display calculi in [68]. The recent result [33] generalizes and abstracts the algorithms for hypersequent calculi in [26, 34] to extract display calculi from suitable Hilbert axioms starting from any "well-behaving" display calculus (*amenable calculus*, see Definition 41).

Section 4.2.1 presents the idea behind the algorithm in [26,34] for single-conclusioned and multiple-conclusioned hypersequent calculi, while Section 4.2.2 its adaptation in [33] for the display calculus.

## 4.2.1. From axioms to hypersequent rules

To simplify the presentation we sketch below the algorithm in [26] (and [34]) taking **HIL** as base calculus.[16] Its key ingredients are: (1) a syntactic classification of Hilbert axioms in the language of intuitionistic logic that accounts for the intuitive difficulty to deal with them proof theoretically (*substructural hierarchy*); (2) the invertibility of some of the rules of **HIL**

---

[16]The algorithm was introduced in [26] for extensions of intuitionistic linear logic without the exponentials (and in [34] for its involutive counterpart MALL).

(3) Ackermann's Lemma [37], below, which permits formulae to 'change the side' of the (hyper)sequent going from the conclusion to the premises.

LEMMA 33 ( [26, 34]). *The rule* $\dfrac{G_1 \quad \cdots \quad G_m}{G \,|\, G' \,|\, A_1, \ldots, A_n \Rightarrow B}$ *(r) derives (and is derivable from) each of the rules*

$$\frac{\vec{G} \quad G \,|\, \Gamma_1 \Rightarrow A_1 \quad \cdots \quad G \,|\, \Gamma_n \Rightarrow A_n}{G \,|\, G' \,|\, \Gamma_1, \ldots, \Gamma_n \Rightarrow B} \qquad \frac{\vec{G} \quad G \,|\, \Sigma, B \Rightarrow \Delta}{G \,|\, G' \,|\, A_1, \ldots, A_n, \Sigma \Rightarrow \Delta}$$

*where $\vec{G}$ abbreviates the premises $G_1 \cdots G_m$, and $\Gamma_1, \ldots, \Gamma_n, \Sigma, \Delta$ are new variables for multisets of formulae (in case of $\Delta$: at most one formula).*

PROOF. Follows using the identity axiom, (*cut*) and (*ec*).  ∎

The substructural hierarchy for intermediate logics is defined by the following grammar: $\mathcal{N}_0, \mathcal{P}_0$ contain the set of atomic formulae.

$$\mathcal{P}_{n+1} ::= \bot \mid \top \mid \mathcal{N}_n \mid \mathcal{P}_{n+1} \wedge \mathcal{P}_{n+1} \mid \mathcal{P}_{n+1} \vee \mathcal{P}_{n+1}$$
$$\mathcal{N}_{n+1} ::= \bot \mid \top \mid P_n \mid \mathcal{N}_{n+1} \wedge \mathcal{N}_{n+1} \mid \mathcal{P}_{n+1} \rightarrow \mathcal{N}_{n+1} \mid \neg \mathcal{P}_{n+1}$$

The classes $\mathcal{P}_n$ and $\mathcal{N}_n$ contain axioms with leading positive and negative connective, respectively. A connective is positive (negative) if its left (right) logical rule is invertible [1]; note that in **HIL**, $\vee$ is positive, $\rightarrow$ is negative and $\wedge$ is both positive and negative.

A graphical representation of the relationships between the classes is the following (the arrows stand for $\subseteq$):
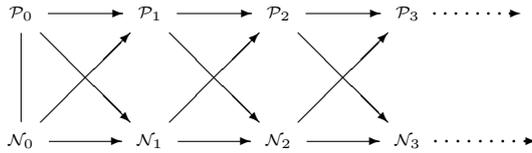


THEOREM 34 ([26]). *Any axiom $\mathcal{A} \in \mathcal{P}_3$ can be transformed into hypersequent structural rules. Adding these rules to **HIL** preserves cut-elimination and yields a calculus sound and complete for the intermediate logic $\mathrm{IL} + \mathcal{A}$.*

REMARK 35. *The substructural hierarchy based on formulas of MALL is defined in a similar way (see [34]) and the theorem above works for a restricted class of axioms $\mathcal{A} \in \mathcal{P}_3$ (acyclic axioms), whose notion in the display calculus setting is described in STEP 4 of Section 4.2.2.*

EXAMPLE 36. *The axioms (lin), (wem) and (em) in Section 2 are within the class $\mathcal{P}_3$. The algorithm contained in the proof of the above Theorem*

*in [26] generates the corresponding rules (com), (lq) and (cl). We describe below the algorithm with the concrete case of*

$$(wem) \ \neg A \vee \neg\neg A.$$

*STEP I. (wem) is equivalent to the hypersequent $G \mid \ \Rightarrow \neg A \mid \ \Rightarrow \neg\neg A$, i.e., **HIL** + (wem) and **HIL** + $G \mid \ \Rightarrow \neg A \mid \ \Rightarrow \neg\neg A$ prove the same sequents. Note that $\neg A, \neg\neg A \in \mathcal{N}_2$.*

*STEP II (transformation into a structural rule). We remove all logical connectives from the obtained hypersequent by applying upwards the invertible logical rules of **HIL** as much as possible, and then using Lemma 33 to change the side of formulae when required. This corresponds to STEP 2 and 3(a) for display calculi (see Section 4.2.2).*

*Back to our example: By applying the invertible rule $(\neg, r)$ twice we obtain the equivalent rule (with no premise) $G \mid A \Rightarrow \ \mid \neg A \Rightarrow$. To get rid of the $\neg$ connective whose introduction rule in the succedent is invertible and in antecedent $(\neg, l)$ is not, we first apply Lemma 33 (thus obtaining the equivalent rule below left[17]) and afterwards $(\neg, r)$ upwards to obtain the equivalent rule (below center).*

*To ensure that the resulting rule is substitutive (cf. Definition 6) we apply Ackermann's Lemma to each formula in the rule conclusion. In our example this means to replace the formula $A$ in the conclusion of the rule (below center) by a multiset variable $\Gamma'$ using Lemma 33, thus obtaining the equivalent structural rule below right.*

$$\frac{G \mid \Gamma \Rightarrow \neg A}{G \mid A \Rightarrow \ \mid \Gamma \Rightarrow} \qquad \frac{G \mid A, \Gamma \Rightarrow}{G \mid A \Rightarrow \ \mid \Gamma \Rightarrow} \qquad \frac{G \mid A, \Gamma \Rightarrow \qquad G \mid \Gamma' \Rightarrow A}{G \mid \Gamma' \Rightarrow \ \mid \Gamma \Rightarrow}$$

*This rule is substitutive and hence preserves cut-elimination. However it fails the subformula property due to the occurrences of $A$ in the premises.*

*STEP III (all possible cuts to get the subformula property). We close the obtained rule under all possible applications of (cut) to its premises. This is what is done in STEP 4 for display calculi (Section 4.2.2). The (lq) rule is then obtained from the rightmost rule in the previous step by applying (cut) to its premises.*

$$\frac{G \mid \Gamma, \Gamma' \Rightarrow}{G \mid \Gamma' \Rightarrow \ \mid \Gamma \Rightarrow} \ (lq)$$

In analogy with display calculi, (for STEP III) the algorithm requires a further condition [34] on the shape of the input axioms when the base hypersequent calculus is multiple-conclusioned.

---

[17]Note that the component $A \Rightarrow$ is the $G'$ in Lemma 33.

The algorithm in [26] is implemented in the PROLOG system *AxiomCalc* available at `http://www.logic.at/people/lara/tinc/webaxiomcalc/`. *AxiomCalc* takes as input any axiom in the language of intuitionistic linear logic without exponentials MAILL and indicates the class to which the axiom belongs. For axioms within the class $\mathcal{P}_3$, the system generates (a paper in pdf format that contains) the equivalent (hyper)sequent rules.

REMARK 37. *The algorithm in [34] can be combined with heuristic methods to extract "good" logical rules from suitable axioms: first extract a logical rule by applying the procedure and keeping a compound formula (that will be the principal formula) in the rule conclusion; if the obtained rule does not preserve cut-elimination, search (manually) for suitable formulae that can be proved only with the use of (cut) and transform these formulae into further structural or logical rules. This idea was used in [34] to rediscover the hypersequent calculus for Łukasiewicz logic presented in Section 2.2 and in [28] to introduce a first cut-free hypersequent calculus for the intermediate logic* $\mathrm{BD}_2$, *whose peculiar axiom belongs to the class* $\mathcal{P}_4$ *(see Example 42). Note that when defining logical rules, the cut-admissibility of the resulting calculus needs either an ad-hoc syntactic proof or suitable semantic methods as in [69].*

### 4.2.2. From axioms to display calculus rules

A structural rule satisfying C1–C8 is said to be *proper*. In [33] the algorithm for hypersequent calculi sketched in the previous section is abstracted and generalized to obtain extensions of display calculi by *proper structural rules*. We sketch the algorithm and illustrate it using the calculus $\delta$Bi-IL for bi-intuitionistic logic Bi-IL (Section 3.3) as a base calculus.

To apply the algorithm we will require that the base calculus satisfies certain conditions (*amenable* calculus, see Definition 41 below).

As in the hypersequent case, the algorithm is based on: (1-2) the invertibility of some of the rules of the base calculus, which determines a suitable syntactic classification of Hilbert axioms; (3) the display calculus analogue, below, of the Ackermann Lemma (Lemma 33) that allows formulae to change the side of the sequent going from the conclusion to the premises:

LEMMA 38. *Let* $\mathcal{C}$ *be an amenable display calculus for a logic* $\mathcal{L}$ *(see Definition 41 below). The following rules are pairwise equivalent in* $\mathcal{C}$

$$\frac{\mathcal{S}}{X \Rightarrow A}\rho_1 \quad \frac{\mathcal{S} \quad A \Rightarrow Z}{X \Rightarrow Z}\rho_2 \qquad \frac{\mathcal{S}}{A \Rightarrow X}\delta_1 \quad \frac{\mathcal{S} \quad Z \Rightarrow A}{Z \Rightarrow X}\delta_2$$

*where $A$ is a formula, $\mathcal{S}$ is a set of display sequents and $Z(\neq X)$ is a structure variable not in $\mathcal{S}$.*

STEP 1. Let $\mathcal{C}$ be an *amenable* display calculus for a logic $\mathcal{L}$. Identify the invertible rules of $\mathcal{C}$.

EXAMPLE 39. *Every logical rule in $\delta$BiIL except $\to l$ and $\leftarrow_d r$ is invertible. Note that we are using the invertible rules for $\land r$ and $\lor l$ (cf. Remark 29).*

STEP 2. Given a formula $A \in$ Bi-IL, repeatedly apply the invertible rules—and display rules where necessary—upwards (i.e., from conclusion to premises) to $\mathbf{I} \Rightarrow A$ to obtain a finite set $\{U_i \Rightarrow V_i\}_{i \in \Omega}$ of sequents. $\mathcal{C} + \{\rho_i\}_{i \in \Omega}$ is a display calculus for $\mathcal{L} + A$ where $\rho_i$ is the 0-premise rule with conclusion $U_i \Rightarrow V_i$ such that all variables therein are treated as schematic. *If each formula appearing in this set is a propositional variable, we say that $A$ is an $\mathcal{I}_1$ formula.*

EXAMPLE 39 (cont.). *Let $lq$ be the formula*[18] *$(p \to \bot) \lor ((p \to \bot) \to \bot)$. Repeatedly apply the invertible rules—and display rules—of $\delta$BiIL upwards to this formula. Note: not all display rule applications are shown.*

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{(\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ p \Rightarrow \mathbf{I}}{(\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ p \Rightarrow \bot} {\bot}r}{\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I}) \Rightarrow p \circ \bot} }{\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I}) \Rightarrow p \to \bot} {\to}r}{\dfrac{(\mathbf{I} \bullet (p \to \bot)) \circ (p \to \bot) \Rightarrow \mathbf{I}}{(\mathbf{I} \bullet (p \to \bot)) \circ (p \to \bot) \Rightarrow \bot}} {\bot}r}{\mathbf{I} \bullet (p \to \bot) \Rightarrow (p \to \bot) \circ \bot}}{\mathbf{I} \bullet (p \to \bot) \Rightarrow ((p \to \bot) \to \bot)} {\to}r}{\mathbf{I} \Rightarrow (p \to \bot) \bullet ((p \to \bot) \to \bot)}}{\mathbf{I} \Rightarrow (p \to \bot) \lor ((p \to \bot) \to \bot)} {\lor}r$$

*So $\delta$BiIL $+ \rho$ is a calculus for BiIl $+ lq$, where $\rho$ is the 0-premise rule with conclusion $(\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ p \Rightarrow \mathbf{I}$. Since the formula $p \to \bot$ cannot be decomposed by invertible rules, $lq$ is not an $\mathcal{I}_1$ formula.*

STEP 3(a). If $A$ is an $\mathcal{I}_1$ formula, then proceed to STEP 4. Otherwise, obtain the rule $\rho_i$ ($i \in \Omega$) from $\{U_i \Rightarrow V_i\}_{i \in \Omega}$ as follows. For each $i$, start with the 0-ary rule with conclusion $U_i \Rightarrow V_i$. Repeatedly using Lemma 38: for each a-part (resp. s-part) occurrence of a formula $B$ in this sequent,

---

[18]Denoting the formula $p \to \bot$ as $\neg p$ we get $\neg p \lor \neg\neg p$.

replace that occurrence with a new structure variable $X_B$ and add a premise $X_B \Rightarrow B$ ($B \Rightarrow X_B$) to ultimately obtain the rule $\rho_i'$. Repeatedly apply the invertible rules—and display rules as required—upwards to each premise of $\rho_i'$ to ultimately obtain the rule $\rho_i$. *If each formula appearing in a premise of $\rho_i$ is a propositional variable, we say that $A$ is an $\mathcal{I}_2$ formula.*

EXAMPLE 39 (cont.). *Display the rightmost $p$ in $(\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ p \Rightarrow \mathbf{I}$ to get $p \Rightarrow (\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ \mathbf{I}$. Using Lemma 38, convert this 0-ary rule to the rule below left. The conclusion of this rule is display equivalent to $p \to \bot \Rightarrow (\mathbf{I} \bullet (X_p \circ \mathbf{I})) \circ \mathbf{I}$. Lemma 38 gives the rule below right.*

$$\frac{X_p \Rightarrow p}{X_p \Rightarrow (\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I})) \circ \mathbf{I}} \qquad \frac{X_p \Rightarrow p \qquad X_{p \to \bot} \Rightarrow p \to \bot}{X_{p \to \bot} \Rightarrow (\mathbf{I} \bullet (X_p \circ \mathbf{I})) \circ \mathbf{I}} \; \rho_1'$$

*Repeatedly applying the invertible rules upwards to the premises of $\rho_1'$ we get*

$$\frac{X_p \Rightarrow p \qquad X_{p \to \bot} \Rightarrow p \circ \mathbf{I}}{X_{p \to \bot} \Rightarrow (\mathbf{I} \bullet (X_p \circ \mathbf{I})) \circ \mathbf{I}} \; \rho_1$$

*Since every formula in the premise of $\rho_1$ is a propositional variable, $lq \in \mathcal{I}_2$.*

STEP 3(b). Suppose that $A$ is an $\mathcal{I}_2$ formula. Then $\mathcal{C} + \{\rho_i\}_{i \in \Omega}$ is a display calculus for the logic $\mathcal{L} + A$ (treat each variable in $\rho_i$ as a schematic formula variable).

It can be verified that $\rho_i$ satisfies C2–C8 (C8 is not relevant for structural rules) so $\mathcal{C} + \{\rho_i\}_{i \in \Omega}$ has cut-elimination. C1 does not hold since there are schematic formulae in the premise but not in the conclusion.

EXAMPLE 39 (cont.). *We have that $\delta$BiIL $+ lq$ is equivalent to the display calculus $\delta$BiIL$+\rho_1$. Since $\rho_1$ satisfies C2-C8 the calculus has cut-elimination.*

STEP 4. (all possible cuts to get the subformula property). Suppose that repeatedly applying (*cut*) on the formulae in the premises of $\rho_i$ for each $i \in \Omega$ ultimately leads to a finite set $\{S_j^i \Rightarrow T_j^i\}_{j \in \Omega_i}$ of sequents. If this is the case then we say that $A$ is *acyclic*. Then it can be shown that $\mathcal{C} + \{\rho_i^*\}_{i \in \Omega}$ is a proper structural rule extension for $\mathcal{L} + A$, where $\rho_i^*$ is obtained by replacing the premises of $\rho_i$ with $\{S_j^i \Rightarrow T_j^i\}_{j \in \Omega_i}$.

EXAMPLE 39 (cont.). *Using the display rules, the premises of $\rho_1$ can be written as $X_p \Rightarrow p$ and $p \Rightarrow X_{p \to \bot} \circ \mathbf{I}$. Applying cut to these premises we get $X_p \Rightarrow X_{p \to \bot} \circ \mathbf{I}$. No further cuts are possible. So $\delta$BiIL $+ \rho_1^*$ is a proper structural rule extension for* BiIL, *where $\rho_1^*$ is the rule below:*

$$\frac{X_p \Rightarrow X_{p \to \bot} \circ \mathbf{I}}{X_{p \to \bot} \Rightarrow (\mathbf{I} \bullet (X_p \circ \mathbf{I})) \circ \mathbf{I}} \; \rho_1^*$$

*Recall that lq was the formula $(p \to \bot) \vee ((p \to \bot) \to \bot)$. Let us verify that $\delta$BiIL $+ \rho_1^*$ derives lq. Since $p \Rightarrow (p \to \bot) \circ \mathbf{I}$ is derivable in $\delta$BiIL, applying $\rho_1^*$ the sequent $p \to \bot \Rightarrow (\mathbf{I} \bullet (p \circ \mathbf{I})) \circ \mathbf{I}$ is derivable. Using the display rules derive $\mathbf{I} \bullet ((p \to \bot) \circ \mathbf{I}) \Rightarrow p \circ \mathbf{I}$. From the derivation in STEP 2 we can see that $\mathbf{I} \Rightarrow lq$ is indeed derivable.*

The main result in [33] is the following.

THEOREM 40. *Let $\mathcal{C}$ be an amenable calculus for $\mathcal{L}$ (Definition 41) and $A$ be a formula in $\mathcal{L}$ belonging to $\mathcal{I}_2$. If $A$ is acyclic then there is a proper structural rule extension of $\mathcal{C}$ for $\mathcal{L} + A$.*

We denote by $\mathcal{L}_\mathcal{C}$ the language whose connectives and constants are those introduced by the logical rules in $\mathcal{C}$.

DEFINITION 41 (amenable calculus). *Let $\mathcal{C}$ be a display calculus satisfying C1–C8. Assume that we have two functions $l$ and $r$ mapping structures into formulae such that $l(A) = r(A) = A$ and suppose for an arbitrary structure $X$ that*

1. *$X \Rightarrow l(X)$ and $r(X) \Rightarrow X$ are derivable in $\mathcal{C}$.*
2. *$X \Rightarrow Y$ derivable implies $l(X) \Rightarrow r(Y)$ is derivable in $\mathcal{C}$.*

*Assume there is a structure constant $\mathbf{I}$ and the following rules are admissible in $\mathcal{C}$ for arbitrary structures $X, Y$ such that the conclusion is well-defined:*

$$\frac{\mathbf{I} \Rightarrow X}{Y \Rightarrow X} \, l\mathbf{I} \qquad\qquad \frac{X \Rightarrow \mathbf{I}}{X \Rightarrow Y} \, r\mathbf{I}$$

*Assume binary logical connectives $\vee, \wedge \in \mathcal{L}_\mathcal{C}$ such that $\cdot \in \{\vee, \wedge\}$ is associative in $\mathcal{C}$ — $A \cdot (B \cdot C) \Rightarrow (A \cdot B) \cdot C$ and $(A \cdot B) \cdot C \Rightarrow A \cdot (B \cdot C)$ are derivable — and commutative in $\mathcal{C}$ — $A \cdot B \Rightarrow B \cdot A$ is derivable. Also suppose:*

$(a)_\vee$ *$A \Rightarrow X$ and $B \Rightarrow X$ implies $A \vee B \Rightarrow X$*

$(b)_\vee$ *$X \Rightarrow A$ implies $X \Rightarrow A \vee B$ for any formula $B$.*

$(a)_\wedge$ *$X \Rightarrow A$ and $X \Rightarrow B$ implies $X \Rightarrow A \wedge B$*

$(b)_\wedge$ *$A \Rightarrow X$ implies $A \wedge B \Rightarrow X$ for any formula $B$.*

*A display calculus satisfying the above conditions is said to be* amenable.

Requiring that $l\mathbf{I}$ and $r\mathbf{I}$ are admissible in $\mathcal{C}$ is weaker than requiring that $\mathcal{C}$ contains weakening rules. Indeed, the rules $l\mathbf{I}$ and $r\mathbf{I}$ are admissible in the Bi-Lambek calculus [53].

EXAMPLE 42. *A calculus $\delta$IL for IL is obtained by deleting the rules $\leftarrow_d l$ and $\leftarrow_d r$ from $\delta$BiIL (see Section 3.3). $\mathcal{I}_2$ axioms with respect to $\delta$IL can be defined using the following grammar: $\mathcal{I}_0 ::=$ prop. variables and*

$$\mathcal{I}_{n+1} ::= \bot \mid \top \mid \mathcal{I}_n \mid \mathcal{I}_n \to \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \wedge \mathcal{I}_{n+1} \mid \mathcal{I}_{n+1} \vee \mathcal{I}_{n+1}$$

*Let $\delta$BiIL $+ \{\rho_i\}_{i\in\Omega}$ be a proper structural rule extension for BiIL $+ A$. If BiIL$+A$ is conservative over IL$+A$, then $\delta$IL$+\{\rho_i\}_{i\in\Omega}$ is a proper structural rule extension for IL $+ A$. In Section 4.2.1 we saw that structural rule extensions for hypersequent calculi can be constructed for intuitionistic axioms within the class $\mathcal{P}_3$ of the substructural hierarchy. Note that $\mathcal{P}_3 \subset \mathcal{I}_2$. It has been shown [33] that $\mathcal{P}_3$ axioms are acyclic for the display calculus. Thus for any $A \in \mathcal{P}_3$ such that $\delta$BiIL $+ A$ is conservative over $\delta$IL $+ A$ we can obtain a display calculus for IL $+ A$. An example of a non-$\mathcal{P}_3$ $\mathcal{I}_1$-formula is the $Bd_2$ axiom $B_2 \vee (B_2 \to (B_1 \vee \neg B_1))$. We compute the structural rule*

$$\frac{Y \Rightarrow X \qquad V \Rightarrow U}{\mathbf{I} \Rightarrow X \bullet (Y \circ (U \bullet (V \circ \mathbf{I})))} \; \rho$$

*BiIL $+ Bd_2$ is known to be conservative over IL$+Bd_2$ (see, e.g. [111]) so we have that $\delta$IL$+\rho$ is a calculus for the intermediate logic $\mathrm{BD}_2$ (i.e., IL$+Bd_2$) with cut-elimination. In contrast no hypersequent structural rule is known for $Bd_2$.*

REMARK 43. *Although the above algorithm is essentially the same as for the hypersequent calculus, the key point is that the greater expressiveness of the display calculus often permits a base calculus with more invertible rules, leading to uniform cut-free display calculi for more axiomatic extensions of the base logic. For example, in $\delta$IL the $\vee r$ rule is invertible while it is not in* **HIL**. *The main reason for this greater expressive power is the fact that the display calculus can operate on a conservative extension of the target logic (see Remark 27).*

## A characterization for tense logics

Another procedure to define proper structural display logic rules for certain axiomatic extensions of the modal logics K and Kt was introduced by Kracht [68]. The class of handled axioms are *primitive tense axioms*. Unlike the previous algorithm, the result here gives sufficient and necessary conditions for presenting a logic via proper structural rule extensions.

Define a *primitive tense axiom* as a formula $A \to B$ where both $A$ and $B$ are constructed from propositional variables and $\top$ using $\{\wedge, \vee, \Diamond, \blacklozenge\}$ and $A$ contains each propositional variable at most once.

THEOREM 44 (Display Theorem). *A tense logic $\mathcal{L}$ is an axiomatic extension of* Kt *by primitive tense axioms iff there is a proper structural rule extension of $\delta$Kt for $\mathcal{L}$.*

Here is the algorithm for computing the structural rule equivalent to the primitive tense formula $A \to B$. Using the valid equivalences $\Diamond(C \vee D) \leftrightarrow \Diamond C \vee \Diamond D$, $\blacklozenge(C \vee D) \leftrightarrow \blacklozenge C \vee \blacklozenge D$ and $(C \vee D) \wedge E \leftrightarrow (C \wedge E) \vee (D \wedge E)$ we can write $A = \bigvee_{i \leq m} C_i$ and $B = \bigvee_{j \leq n} D_j$ where every $C_i$ and $D_j$ is built up from propositional variables and $\top$ using $\{\wedge, \blacklozenge, \Diamond\}$. Now $\bigvee_{i \leq m} C_i \to \bigvee_{j \leq n} D_j$ is a theorem of Kt iff $C_i \to \bigvee_{j \leq n} D_j$ is a theorem of Kt for all $i$, $1 \leq i \leq m$. Translate each $C_i \to \bigvee_{j \leq n} D_j$ into the structural rule $\rho_i$ (below left) by replacing the $\to$ with the symbol $\Rightarrow$, and using the map $\sigma$ (below right), where $X_p$ is a schematic structure variable that is uniquely assigned to the propositional variable $p$ and $X$ is a new structure variable.

$$\frac{\sigma(D_1) \Rightarrow X \quad \ldots \quad \sigma(D_n) \Rightarrow X}{\sigma(C_i) \Rightarrow X} \, \rho_i \qquad\qquad \begin{aligned} &\sigma(\top) = \mathbf{I} \\ &\sigma(p) = X_p \\ &\sigma(A \wedge B) = \sigma(A) \circ \sigma(B) \\ &\sigma(\blacklozenge B) = \bullet\sigma(B) \\ &\sigma(\Diamond B) = * \bullet *\sigma(B) \end{aligned}$$

Then $\delta$Kt $+ \{\rho_i\}_{i \leq m}$ is a proper structural rule extension for Kt $+ A \to B$.

EXAMPLE 45. *Consider the axioms $\Box p \to \Box\Box p$ and $\Diamond\Box p \to \Box\Diamond p$ for transitivity and connectedness. It may be checked that these are equivalent to the primitive tense axioms $\Diamond\Diamond p \to \Diamond p$ and $\blacklozenge\Diamond p \to \Diamond\blacklozenge p$. Using the map $\sigma$ above, we immediately obtain the structural rules below such that $\delta$Kt $+ \rho_1$ and $\delta$Kt $+ \rho_2$ are proper structural rule extensions, respectively, for the logics* Kt $+ \Box p \to \Box\Box p$ *and* Kt $+ \Diamond\Box p \to \Box\Diamond p$.

$$\frac{(* \bullet *)(* \bullet *)Z \Rightarrow X}{(* \bullet *)Z \Rightarrow X} \, \rho_1 \qquad\qquad \frac{(* \bullet *) \bullet Z \Rightarrow X}{\bullet(* \bullet *)Z \Rightarrow X} \, \rho_2$$

*Using cut-elimination and conservativity, $\delta$K $+ \rho_1$ and $\delta$K $+ \rho_2$ are calculi for the modal logics* K $+ \Box p \to \Box\Box p$ *and* K $+ \Diamond\Box p \to \Box\Diamond p$, *respectively.*

REMARK 46. *$\delta$Kt is an amenable calculus and each primitive tense axiom $A \to B$ is an acyclic $\mathcal{I}_2$ formula (due to the condtion on the uniqueness of propositional variables in $A$). Hence the algorithm in [33] provides an alternative proof of the "only if" direction of Theorem 44.*

Kracht's method of extracting structural rules from (primitive tense) axioms is very different to the method [33] presented before and leads to syntactically different structural rules. His method relies on the ability to transform an axiom into a primitive tense axiom.

### 4.3.  Some brief philosophical considerations

Gentzen-style proof systems form the basis of the proof-theoretic semantics of the logical operations, see [94, 110]. The meaning-theoretical impact of natural deduction was elaborated by Prawitz [85], who realized the philosophical significance of Gentzen's remark that the introduction rules may be seen as definitions of the logical operations in question, whereas the elimination rules, in the final analysis, can be derived from the introduction rules. The idea is that the formula that contains the logical operation to be eliminated as its main logical operation may be used only as what it *means* on the basis of the introduction of that operation. Dummett [42] introduced a notion of proof-theoretic harmony in order to characterize the relation between the introduction and elimination rules, whereas Prawitz defined an "inversion principle". According to Prawitz, it is the inversion principle that guarantees that the elimination rules are *semantically justified* by their introduction rules. Moreover, for Prawitz natural deduction is the most convincing framework for defining a proof-theoretic semantics.

Although Gentzen [50] developed the sequent calculus for purely formal reasons, namely to prove cut-elimination and the consistency of first-order arithmetic, sequent calculi have also been proposed as a type of proof systems suitable for the development of proof-theoretic semantics. Schroeder-Heister [93, p. 237], for example, considers the sequent calculus as a more adequate formal model of hypothetical reasoning, "because it does more justice to the notion of assumption than does natural deduction," and Standefer [95, p. 288] explains that "because of the importance of structure for the meaning of the logical connectives, the presence of structural rules in consecution calculi counts heavily in their favor." An influential early source for suggesting sequent calculi as a framework for meaning-theoretical analyses is [60]; more recent references include [16] and [80]. With respect to such meaning-theoretical considerations, a divergence may be noted between hypersequent and display calculi. Whereas hypersequents have been introduced mainly as a formal device for obtaining cut-free sequent calculi for specific logics (see [2] for RM and [84] for S5), display sequents have been motivated by formal *and* meaning-theoretical reasons. When Belnap introduced display calculi, he was interested in a proof-theoretic framework that is flexible enough to combine logical operations from different families of operations: classical, relevant, intuitionistic, modal, etc. Moreover, he was interested in a general cut-elimination theorem that covers a wide spectrum of logics as compared with piecemeal proofs of cut-admissibility for these systems. In [105] it is argued that the structural connectives of display calculi can

be seen as context-dependent data structuring operations. The unary $\bullet$, for instance, may bee seen as marking a structure in its scope as modal. The display and structural rules contribute to what Belnap [14] calls a deductive context. Display calculi provide very rich deductive contexts in which the introduction rules may be seen as meaning-constitutive. Different contexts, in particular, different structural rules effect differences in meaning for a broad range of logical operations, including modal operators. A notion of proof-theoretic harmony between left and right introduction rules in display calculi has been suggested in [95]. See, e.g., [5] for a discussion in the context of the sequent calculus. It remains to be investigated to which extent a proof-theoretic semantics based on hypersequent calculi can be developed.

## 5.   Other generalizations of the sequent calculus

In addition to hypersequent and display calculi many other extensions of the sequent calculus have been introduced. Two main approaches to generalizing Gentzen's notion of a sequent can be distinguished: via syntax and via semantics. In the syntactic approach sequents are generalized by allowing extra structural connectives in addition to the sequent comma; in the semantic approach, the semantic language is an explicit part of the syntax in sequents and rules. We conclude the paper by mentioning some well-known frameworks that use these two approaches:

**Syntactic Frameworks**
**Dunn-Mints Systems** have been developed independently by Dunn [43] and Mints [77] (see also [79]) to provide a cut-free formulation of logics lacking the weakening rule but satisfying the distributivity axiom $(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))$. The guiding idea behind these systems is to consider "sequents" which contain metalogical symbols for both the additive and multiplicative conjunction (see e.g. [100] for this terminology). **Deep-inference calculi:** In all the formalisms mentioned so far, rules only see the root of formulae. The Calculus of Structures (CoS) is a framework that uses (*deep inference*) rules which, instead, can rewrite at any position in the formula tree, see [59]. E.g., the CoS rule corresponding to $(\vee, l)$ in **LK** is

$$\frac{s((\Gamma \vee A) \wedge (B \vee \Delta))}{s(\Gamma \vee (A \wedge B) \vee \Delta)}$$

where $s$ is an additional schematic variable indicating that the inference rule can be applied anywhere. CoS works best for logics with an involutive

negation and has been successfully applied to many logics including classical logic, linear logic, intuitionistic logic and some modal logics, see e.g. [56, 59] and the web page [58]. Some direct **generalizations of hypersequents** are noncommutative hypersequents [24], introduced to define cut-free calculi for the intermediate logics $BD_n$ with bounded depth Kripke models, the hypersequents in [99] for Corsi's logic in which eigenvariables form an explicit part of the syntactic structure (cf. Section 2.3), and tree-hypersequents [82] that work for a large class of modal logics. The latter calculi are a notational variant [55] of (shallow) **nested sequents** [20, 66], which extend ordinary sequents by permitting a nesting of sequents. Connections between deep nested sequents and display calculi have been explored in [36, 54].

### Semantic frameworks[19]

**Labelled calculi** internalizing Kripke semantics in the sequents are the most developed systems within the semantic approach. Modular treatments of modal and substructural logics possessing a natural relational semantics easily follow in this framework [46, 48, 78, 102]. Also, **Rasiowa-Sikorski systems** were successfully used to define analytic calculi for various classes of logics, see e.g. the survey in [67]. (First-order) finite-valued logics are instead naturally characterized by calculi in which labels are interpreted as sets of truth values [9, 11] or by $n$-sided sequents, each one representing a truth value, e.g., [10, 92] or sets of truth values [61]. **Sequent-of-relations** [8] (see also [32]), whose basic objects are finite sets of *components* of the form $A < B$ or $A \leq B$ for arbitrary formulae $A$ and $B$, capture *projective logics*, of which propositional finite-valued logics are particular instances. When these components contain arbitrary sets of formulae, the resulting proof system is that of **relational hypersequents**, introduced in [23] to define uniform calculi (the logical rules are the same) for the three infinite-valued logics formalizing Fuzzy Logic [62]: Gödel, Łukasiewicz and product logic.

---

[19]The distinction we are drawing between syntactic and semantic frameworks is neither categorical nor entirely uncontroversial. Tree-hypersequents encode a *semantic* accessibility relation, $n$-sided sequents represent *truth values* in a syntactical way as places of derivability statements, and the *semantic tableaux* system for $CL^{fo}$ is very closely related to Gentzen's **LK**. A recent discussion of methodological issues related to the use of display sequents, tree-hypersequents and labelled sequents in modal logic can be found in [83].

# References

[1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *J. Logic Comput.*, 2(3):297–347, 1992.

[2] A. Avron. A constructive analysis of RM. *J. of Symbolic Logic*, 52(4):939–951, 1987.

[3] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(3–4):225–248, 1991.

[4] A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: from foundations to applications (Staffordshire, 1993)*, Oxford Sci. Publ., pages 1–32. Oxford Univ. Press, New York, 1996.

[5] A. Avron and I. Lev. Non-deterministic multi-valued structures. *J. Logic Comput.*, 15:241–261, 2005.

[6] M. Baaz, A. Ciabattoni, and C. G. Fermüller. Hypersequent calculi for Gödel logics: A survey. *J. Logic Comput.*, 13:1–27, 2003.

[7] M. Baaz, A. Ciabattoni, and F. Montagna. Analytic calculi for monoidal t-norm based logic. *Fundamenta Informaticae*, 59(4):315–332, 2004.

[8] M. Baaz and C. G. Fermüller. Analytic calculi for projective logics. In *Tableaux '99. LNAI*, volume 1617, pages 36–50. Springer, 1999.

[9] M. Baaz, C.G. Fermüller, G. Salzer, and R. Zach. Labeled calculi and finite-valued logics. *Studia Logica*, 61(1):7–33, 1998.

[10] M. Baaz, C.G. Fermüller, and R. Zach. Elimination of cuts in first-order many-valued logics. *J. of Inf. Proc. and Cybernetics*, 29:333–355, 1994.

[11] M. Baaz, O. Lahav, and A. Zamansky. Finite-valued semantics for canonical labelled calculi. *J. of Automated Reasoning*, 2013.

[12] M. Baaz and R. Zach. Hypersequents and the proof theory of intuitionistic fuzzy logic. In *CSL'00. LNCS*, pages 187–201. Springer, 2000.

[13] P. Baldi, A. Ciabattoni, and L. Spendier. Standard completeness for extensions of MTL: an automated approach. In *WOLLIC 2012*, volume 7456 of *LNCS*, pages 154–167. Springer, 2012.

[14] N. D. Belnap, Jr. Tonk, plonk and plik. *Analysis*, 22:130–134, 1962.

[15] N. D. Belnap, Jr. Display logic. *J. Philos. Logic*, 11(4):375–417, 1982.

[16] N. D. Belnap, Jr. The display problem. In H. Wansing, editor, *Proof Theory of Modal Logic*, pages 79–92. Kluwer, Dordrecht, 1996.

[17] P. Blackburn, M. de Rijke, and I. Venema. *Modal logic*, volume 53 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, Cambridge, 2001.

[18] J. Brotherston. Bunched logics displayed. *Studia Logica*, 100(6):1223–1254, 2012.

[19] J. Brotherston and R. Goré. Craig interpolation in displayable logics. In *Tableaux 2011*, volume 6793 of *LNCS*, pages 88–103. Springer, 2011.

[20] K. Brünnler. Deep sequent systems for modal logic. In *Advances in modal logic. Vol. 6*, pages 107–119. Coll. Publ., London, 2006.

[21] A. Ciabattoni. Automated generation of analytic calculi for logics with linearity. In *CSL04. LNCS*, volume 3210, pages 503–517. Springer, 2004.

[22] A. Ciabattoni. A proof-theoretical investigation of global intuitionistic (fuzzy) logic. *Archive of mathematical Logic*, 44:435–457, 2005.

[23] A. Ciabattoni, C. G. Fermüller, and G. Metcalfe. Uniform rules and dialogue games for fuzzy logics. In *LPAR 2004. LNCS*, volume 3452, pages 496–510, 2004.

[24] A. Ciabattoni and M. Ferrari. Hypertableau and path-hypertableau calculi for some families of intermediate logics. In *Tableaux 2000. LNAI*, volume 1847, pages 160–175, 2000.

[25] A. Ciabattoni, D. Gabbay, and N. Olivetti. Cut-free proof systems for logics of weak excluded middle. *Soft Computing*, 2(4):147–156, 1999.

[26] A. Ciabattoni, N. Galatos, and K. Terui. From axioms to analytic rules in nonclassical logics. In *LICS 2008*, pages 229–240, 2008.

[27] A. Ciabattoni, N. Galatos, and K. Terui. Algebraic proof theory ii: hypersequents and hypercompletions. submitted. 2014.

[28] A. Ciabattoni, P. Maffezioli, and L. Spendier. Hypersequent and labelled calculi for intermediate logics. In *Tableaux 2013. LNCS*, volume 8123, pages 81–96. Springer, 2013.

[29] A. Ciabattoni and G. Metcalfe. Bounded Łukasiewicz logics. In *Tableaux 2003. LNCS*, volume 2796, pages 32–47. Springer, 2003.

[30] A. Ciabattoni and G. Metcalfe. Density elimination. *Theor. Comput. Sci.*, 403(2-3):328–346, 2008.

[31] A. Ciabattoni, G. Metcalfe, and F. Montagna. Algebraic and proof-theoretic characterizations of truth stressers for MTL and its extensions. *Fuzzy Sets and Systems*, 161(3):369–389, 2010.

[32] A. Ciabattoni and F. Montagna. Proof theory for locally finite many-valued logics: Semi-projective logics. *Theor. Comput. Sci.*, 480:26–42, 2013.

[33] A. Ciabattoni and R. Ramanayake. Structural rule extensions of display calculi: a general recipe. In *WOLLIC 2013*, volume 8071 of *LNCS*, pages 81–95. Springer, 2013.

[34] A. Ciabattoni, L. Strassburger, and K. Terui. Expanding the realm of systematic proof theory. In *CSL 2009. LNCS*, pages 163–178. Springer, 2009.

[35] R. Horčík. Algebraic semantics: Semilinear FL-algebras. In *Handbook of Mathematical Fuzzy Logic, vol 1*, volume 37 of *Studies in Logic, Mathematical Logic and Foundations*, pages 175–214. College Publications, London, 2011.

[36] R. Clouston, R. Goré, and A. Tiu. Annotation-free sequent calculi for full intuitionistic linear logic. In *CSL 2013*, LNCS, pages 197–214. Springer, 2013.

[37] W. Conradie and A. Palmigiano. Algorithmic correspondence and canonicity for distributive modal logic. *Annals of Pure and Applied Logic*, 163(3):338–376, 2012.

[38] G. Corsi. A cut-free calculus for Dummett's LC quantified. *Mathematical Logic Quarterly*, 35(4):289–301, 1989.

[39] J. E. Dawson and R. Goré. Formalised cut admissibility for display logic. In *Theorem proving in higher order logics*, volume 2410 of *LNCS*, pages 131–147. Springer, Berlin, 2002.

[40] J. E. Dawson and R. Goré. A new machine-checked proof of strong normalisation for display logic. In *Electronic Notes in Theoretical Computer Science*. Elsevier, 2002.

[41] S. Demri and R. Goré. Display calculi for nominal tense logics. *J. Logic Comput.*, 12(6):993–1016, 2002.

[42] M. Dummet. *Frege. Philosophy of Language.* Harper & Row, New York, 1973.

[43] J.M. Dunn. A 'Gentzen' system for positive relevant implication. *J. of Symbolic Logic*, 38:356–357, 1974. (Abstract).

[44] H. Enderton. *A mathematical introduction to logic*. Academic Press, New York, 1972.

[45] F. Esteva and L. Godo. Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124:271–288, 2001.

[46] M. Fitting. *Proof methods for modal and intuitionistic logics*, volume 169 of *Synthese Library*. D. Reidel Publishing Co., Dordrecht, 1983.

[47] A. Kurz G. Greco and A. Palmigiani. Dynamic epistemic logic displayed. In *Logic, Rationality, and Interaction*, volume 8196 of *LNCS*, pages 135–148. Springer, 2013.

[48] D. Gabbay. *Labelled Deductive Systems*, volume 1—Foundations. Oxford University Press, 1996.

[49] N. Galatos, P. Jipsen, T. Kowalski, and H. Ono. *Residuated lattices: an algebraic glimpse at substructural logics*, volume 151 of *Studies in Logic and the Foundations of Mathematics*. Elsevier B. V., Amsterdam, 2007.

[50] G. Gentzen. Untersuchungen über das logische schließen. *Mathematische Zeitschrift*, 39:176–210, 405–431, 1934/35. English translation in: *American Philosophical Quarterly* 1 (1964), 288–306 and *American Philosophical Quarterly* 2 (1965), 204–218, as well as in: *The Collected Papers of Gerhard Gentzen*, (ed. M.E. Szabo), Amsterdam, North Holland (1969), 68–131.

[51] R. Goldblatt. *Topoi. The Categorical Analysis of Logic*. North-Holland, Amsterdam, 1979.

[52] R. Goré. Gaggles, Gentzen and Galois: how to display your favourite substructural logic. *Log. J. IGPL*, 6(5):669–694, 1998.

[53] R. Goré. Substructural logics on display. *Log. J. IGPL*, 6(3):451–504, 1998.

[54] R. Goré, L. Postniece, and A. Tiu. On the correspondence between display postulates and deep inference in nested sequent calculi for tense logics. *Log. Methods Comput. Sci.*, 7(2):2:8, 38, 2011.

[55] R. Goré and R. Ramanayake. Labelled tree sequents, tree hypersequents and nested (deep) sequents. In *Advances in modal logic. Volume 9*. College Publications, London, 2012.

[56] R. Goré and A. Tiu. Classical modal display logic in the calculus of structures and minimal cut-free deep inference calculi for S5. *J. Logic Comput.*, 17(4):767–794, 2007.

[57] N. Grazl. Sequent calculi for multi-modal logic with interaction. In D. Grossi, O. Roy, and H. Huang, editors, *Logic, Rationality, and Interaction*, volume 8196 of *Lecture Notes in Computer Science*, pages 124–134. Springer Berlin Heidelberg, 2013.

[58] A. Guglielmi. Deep inference. `http://alessio.guglielmi.name/res/cos/index.html`. Accessed: 2013-10-14.

[59] A. Guglielmi. A system of interaction and structure. *TOCL*, 8(1), 2007.

[60] I. Hacking. What is logic? *J. Phil.*, 76:285–319, 1979.

[61] R. Hähnle. *Automated Deduction in Multiple-valued Logics*. Oxford University Press, 1993.

[62] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, Dordrecht, 1998.

[63] R. Iemhoff and G. Metcalfe. Hypersequent systems for the admissible rules of modal and intermediate logics. In *LNCS*, volume 5407, pages 230–245. Springer, 2009.

[64] R. Iemhoff and G. Metcalfe. Proof theory for admissible rules. *Annals of Pure and Applied Logic*, 159(1-2):171–186, 2009.

[65] A. Indrzejczak. Cut-free hypersequent calculus for S4.3. *Bull. Sect. Logic Univ. Łódź*, 41(1-2):89–104, 2012.

[66] R. Kashima. Cut-free sequent calculi for some tense logics. *Studia Logica*, 53(1):119–135, 1994.

[67] B. Konikowska. Rasiowa-Sikorski Deduction Systems in CS applications. *Theoretical Computer Science*, 286:323–366, 2002.

[68] M. Kracht. Power and weakness of the modal display calculus. In *Proof theory of modal logic (Hamburg, 1993)*, volume 2 of *Appl. Log. Ser.*, pages 93–121. Kluwer Acad. Publ., Dordrecht, 1996.

[69] O. Lahav. From frame properties to hypersequent rules in modal logics. In *LICS 2013, IEEE*, pages 408–417, 2013.

[70] D. Leivant. Quantifiers as modal operators. *Studia Logica*, 39:145–158, 1980.

[71] B. Lellmann and D. Pattinson. Correspondence between modal Hilbert axioms and sequent rules with an application to S5. In *Tableaux 2013. LNCS*, volume 8123, pages 219–233. Springer, 2013.

[72] G. Metcalfe and F. Montagna. Substructural fuzzy logics. *J. of Symbolic Logic*, 72(3):834–864, 2007.

[73] G. Metcalfe and N. Olivetti. Towards a proof theory of Gödel modal logics. *Logical Methods in Computer Science*, 7(2):1–27, 2011.

[74] G. Metcalfe, N. Olivetti, and D. Gabbay. Analytic proof calculi for product logics. *Archive for Mathematical Logic*, 43(7):859–889, 2004.

[75] G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and hypersequent calculi for abelian and Łukasiewicz logics. *ACM Transactions on Computational Logic*, 6(3):578–613, 2005.

[76] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics*, volume 39 of *Springer Series in Applied Logic*. Springer, 2009.

[77] G. Mints. Cut-elimination theorem in relevant logics. *The Journal of Soviet Mathematics*, 6:422–428, 1976.

[78] S. Negri. Proof analysis in modal logic. *J. Philos. Logic*, 34(5-6):507–544, 2005.

[79] F. Paoli. *Substructural Logics: A Primer*. Kluwer Academic Publishers, Dordrecht, 2002.

[80] F. Paoli. Implicational paradoxes and the meaning of logical constants. *Australasian Journal of Philosophy*, 85:553–579, 2007.

[81] F. Poggiolesi. A cut-free simple sequent calculus for modal logic S5. *The Review of Symbolic Logic*, 1(1):3–15, 2008.

[82] F. Poggiolesi. *Gentzen Calculi for Modal Propositional Logic*. Trends in Logic, Springer, 2010.

[83] F. Poggiolesi and G. Restall. Interpreting and applying proof theories for modal logic. In G. Restall and G. Russell, editors, *New Waves in Philosophical Logic*, pages 39–62. Palgrave Macmillan, Basingstoke, 2012.

[84] G. Pottinger. Uniform, cut-free formulations of T, S4 and S5 (abstract). *J. of*

*Symbolic Logic*, 48(3):900, 1983.

[85] D. Prawitz. *Natural Deduction. A Proof-theoretical Study.* Almqvist and Wiksell, Stockholm, 1965.

[86] R. Ramanayake. Embedding the hypersequent calculus in the display calculus. submitted. 2013.

[87] C. Rauszer. A formalization of the propositional calculus of $H - B$ logic. *Studia Logica*, 33:23–34, 1974.

[88] G. Restall. Display logic and gaggle theory. *Rep. Math. Logic*, (29):133–146, 1995.

[89] G. Restall. Displaying and deciding substructural logics. I. Logics with contraposition. *J. Philos. Logic*, 27(2):179–216, 1998.

[90] G. Restall. *An Introduction to Substructural Logics.* Routledge, London, 1999.

[91] G. Restall. A cut-free sequent system for two-dimensional modal logic, and why it matters. *Ann. Pure Appl. Logic*, 163(11):1611–1623, 2012.

[92] G. Rousseau. Sequents in many valued logic I, and II. *Fundamenta Mathematicæ*, LX, LXVII:23–33, 125–131, 1967, 1970.

[93] P. Schroeder-Heister. Sequent calculi and bidirectional natural deduction: On the proper basis of proof-theoretic semantics. In M. Peliš, editor, *Logica Yearbook 2008*, pages 237–251. College Publications, London, 2008.

[94] P. Schroeder-Heister. Proof-theoretic semantics. *The Stanford Encyclopedia of Philosophy* Edward N. Zalta (ed.), Spring 2013 Edition.

[95] S. Standefer. Philosophical aspects of display logic. In M. Peliš, editor, *Logica Yearbook 2009*, pages 283–295. College Publications, London, 2009.

[96] G. Takeuti. *Proof theory*, volume 81 of *Studies in Logic and the Foundations of Mathematics.* North Holland, Amsterdam, 1987.

[97] G. Takeuti and T. Titani. Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *J. of Symbolic Logic*, 49(3):851–866, 1984.

[98] A. Tiu. A system of interaction and structure. II. The need for deep inference. *Log. Methods Comput. Sci.*, 2(2):2:4, 24, 2006.

[99] A. Tiu. A hypersequent system for Gödel-Dummett logic with non-constant domains. In *Tableaux 2011. LNAI*, pages 248–262. Springer, 2011.

[100] A. S. Troelstra and H. Schwichtenberg. *Basic proof theory*, volume 43 of *Cambridge Tracts in Theoretical Computer Science.* Cambridge University Press, Cambridge, second edition, 2000.

[101] J. van Benthem. Modal foundations of predicate logic. *Log. J. IGPL*, 5:259–286, 1997.

[102] L. Viganò. *Labelled non-classical logics.* Kluwer Academic Publishers, Dordrecht, 2000. With a foreword by Dov M. Gabbay.

[103] H. Wansing. Sequent calculi for normal modal propositional logics. *J. Logic Comput.*, 4(2):125–142, 1994.

[104] H. Wansing. Modal tableaux based on residuation. *J. Logic Comput.*, (7):719–731, 1997.

[105] H. Wansing. *Displaying Modal Logic.* Springer, Trends in Logic, 1998.

[106] H. Wansing. Translation of hypersequents into display sequents. *Log. J. IGPL*, 6(5):719–733, 1998.

[107] H. Wansing. Predicate logics on display. *Studia Logica*, 62(1):49–75, 1999.

[108] H. Wansing. Sequent systems for modal logics. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 8, pages 61–145. Kluwer, 2002.

[109] H. Wansing. Constructive negation, implication, and co-implication. *J. Appl. Non-Classical Logics*, 18(2-3):341–364, 2008.

[110] H. Wansing. Prawitz, proofs, and meaning. In H. Wansing, editor, *Dag Prawitz on proofs and meaning*. Springer, to appear 2014.

[111] F. Wolter. On logics with coimplication. *J. Philos. Logic*, 27(4):353–387, 1998.

AGATA CIABATTONI
Technische Universität Wien
Department of Computer Languages
Favoritenstraße 9-11
A-1040 Wien, Austria
agata@logic.at

REVANTHA RAMANAYAKE
Technische Universität Wien
Department of Computer Languages
Favoritenstraße 9-11
A-1040 Wien, Austria
revantha@logic.at

HEINRICH WANSING
Ruhr University Bochum
Department of Philosophy II
Universitätsstraße 150
D-44780 Bochum, Germany
Heinrich.Wansing@rub.de