5

Algebras and General Frames

In this chapter we develop an *algebraic* semantics for modal logic. The basic idea is to extend the algebraic treatment of classical propositional logic (which uses *boolean algebras*) to modal logic. The algebras employed to do this are called *boolean algebras with operators (*BAO*s)*. The boolean part handles the underlying propositional logic, the additional operators handle the modalities.

But why algebraize modal logic? There are two main reasons. First, the algebraic perspective allows us to bring powerful new techniques to bear on modal-logical problems. Second, the algebraic semantics turns out to be better behaved than frame-based semantics: we will be able to prove an algebraic completeness result for *every* normal modal logic. As our discussion of incompleteness in Section 4.4 makes clear, no analogous result holds for frames.

This chapter has three main parts. The first, consisting of the first three sections, introduces the algebraic approach: we survey the basic ideas in the setting of classical propositional logic, extend them to modal logic, and prove the Jónsson-Tarski Theorem. The second part, which consists of the fourth section, introduces *duality theory*, the study of correspondences between the universe of algebras and the universe of frames. The last part (the only part on the advanced track), is devoted to *general frames*. These turn out to be set-theoretic representations of boolean algebras with operators, and we examine their properties in detail, and use them to prove the Sahlqvist Completeness Theorem. Background information on universal algebra can be found in Appendix B.

Chapter guide

Section 5.1: Logic as Algebra (Basic track). What is algebraic logic? This section provides some preliminary answers by examining the relationship between propositional logic and boolean algebras.

Section 5.2: Algebraizing Modal Logic (Basic track). To algebraize modal logic, we introduce *boolean algebras with operators* (BAOs). We discuss BAOs from a semantic perspective (introducing an important class of BAOs called *complex algebras*), and from a syntactic perspective (we use *Lindenbaum-Tarski algebras* to obtain abstract BAOs from normal modal logics).

Section 5.3: The Jónsson-Tarski Theorem (Basic track). Here we prove the theorem underlying algebraic approaches to modal completeness theory. First we learn how to construct a frame from an algebra by forming the *ultrafilter frame*. By turning this frame back into a complex algebra, we obtain the *canonical embedding algebra*. We then prove the Jónsson-Tarski Theorem: every boolean algebra with operators can be embedded in its canonical embedding algebra.

Section 5.4: Duality Theory (Basic track). Frames are inter-related by bounded morphisms, generated subframes, and disjoint union. Boolean algebras with operators are inter-related by homomorphisms, subalgebras, and direct products. Modal duality theory studies the relationship between these two mathematical universes. Two applications are given, one of which is an algebraic proof of the Goldblatt-Thomason Theorem.

Section 5.5: General Frames (Advanced track). We (re)introduce general frames and study them in detail, focusing on the relationship between general frames, frames, and boolean algebras with operators. We conclude with a brief discussion of some important topological aspects of general frames.

Section 5.6: Persistence (Advanced track). In this section we introduce a natural generalization of the notion of canonicity encountered in Chapter 4: *persistence*. We use it to prove the *Sahlqvist Completeness Theorem*.

5.1 Logic as Algebra

What do algebra and logic have in common? And why bring algebra into the study of logic? This section provides some preliminary answers: we show that algebra and logic share key ideas, and analyze classical propositional logic algebraically. Along the way we will meet a number of important concepts (notably formula algebras, the algebra of truth values, set algebras, abstract boolean algebras, and Lindenbaum-Tarski algebras) and results (notably the Stone Representation Theorem), but far more important is the overall picture. Algebraic logic offers a natural way of re-thinking many basic logical issues, but it is important not to miss the wood for the trees. The bird's eye view offered here should help guide the reader through the more detailed modal investigations that follow.

Algebra as logic

Most school children learn how to manipulate simple algebraic equations. Given the expression $(x + 3)(x + 1)$, they learn how to multiply these factors to form

$x^2 + 4x + 3$, and (somewhat later) study methods for doing the reverse (that is, for decomposing quadratics into factors).

Such algebraic manipulations are essentially logical. For a start, we have a well defined syntax: we manipulate *equations* between *terms*. This syntax is rarely explicitly stated, but most students (building on the analogy with basic arithmetic) swiftly learn how to build legitimate terms using numerals, variables such as x, y, and z, and $+$, $\cdot$, and $-$. Moreover, they learn the rules which govern this symbol manipulation process: replacing equals by equals, doing the same thing to both sides of an equation, appealing to commutativity, associativity and distributivity to simplify and rearrange expressions. High-school algebra is a form of proof theory.

But there is also a *semantic* perspective on basic algebra, though this usually only becomes clear later. As students learn more about mathematics, they realize that the familiar 'laws' don't hold for all mathematical objects: for example, matrix multiplication is not commutative. Gradually the student grasps that variables need not be viewed as standing for numbers: they can be viewed as standing for other objects as well. Eventually the semantic perspective comes into focus: there are various kinds of *algebras* (that is, sets equipped with collections of functions, or *operations*, which satisfy certain properties), and *terms denote elements in algebras*. Moreover, an equation such as $x \cdot y = y \cdot x$ is not a sacrosanct law: it's simply a property that holds for some algebras and not for others.

So algebra has a syntactic dimension (terms and equations) and a semantic dimension (sets equipped with a collection of operations). And in fact there is a tight connection between the proof theory algebra offers and its semantics. In Appendix B we give a standard derivation system for *equational* logic (that is, a standard set of rules for manipulating equations) and state a fundamental result due to Birkhoff: the system is strongly sound and complete with respect to the standard algebraic semantics. Algebra really can be viewed as logic.

But logic can also be viewed as algebra. We will now illustrate this by examining classical propositional logic algebraically. Our discussion is based around three main ideas: the algebraization of propositional semantics in the class of set algebras; the algebraization of propositional axiomatics in the class of abstract boolean algebras; and how the Stone Representation Theorem links these approaches.

Algebraizing propositional semantics

Consider any propositional formula, say $(p \vee q) \wedge (p \vee r)$. The most striking thing about propositional formulas (as opposed to first-order formulas) is their syntactic simplicity. In particular, there is no variable binding — all we have is a collection of atomic symbols (p, q, r, and so on) that are combined into more complex expressions using the symbols $\bot$, $\top$, $\neg$, $\vee$ and $\wedge$. Recall that we take $\bot$, $\neg$ and $\vee$ as the primitive symbols, treating the others as abbreviations.

Now, as the terminology 'propositional variable' suggests, we think of p, q, and r as symbols denoting entities called propositions, abstract bearers of information. So what do $\bot$, $\top$, $\neg$, $\vee$ and $\wedge$ denote? Fairly obviously: ways of combining propositions, or operations on propositions. More precisely, $\vee$ and $\wedge$ must denote binary operations on propositions (let's call these operations $+$ and $\cdot$ respectively), $\neg$ must denote a unary operation on propositions (let's call it $-$), while $\bot$ and $\top$ denote special nullary operations on propositions (that is, they are the names of two special propositions: let's call them 0 and 1 respectively). In short, we have worked our way towards the idea that *formulas can be seen as terms denoting propositions*.

But which kinds of algebras are relevant? Here's a first step towards an answer.

Definition 5.1 Let $Bool$ be the algebraic similarity type having one constant (or nullary function symbol) $\bot$, one unary function symbol $\neg$, and one binary function symbol $\vee$. Given a set of propositional variables Φ, $Form(\Phi)$ is the set of $Bool$-terms in Φ; this set is identical to the collection of propositional formulas in Φ.

Algebras of type $Bool$ are usually presented as 4-tuples $\mathfrak{A} = (A, +, -, 0)$. We make heavy use of the standard abbreviations $\cdot$ and 1. That is, $a \cdot b$ is shorthand for $-(-a + -b)$, and 1 is shorthand for -0. $\dashv$

But this only takes us part of the way. There are many different algebras of this similarity type — and we're only interested in algebras which can plausibly be viewed as algebras of propositions. So let's design such an algebra. Propositional logic is about truth and falsehood, so lets take the set $2 = \{0, 1\}$ as the set A underlying the algebra; we think of '0' as the truth value *false*, and '1' as the value *true*. But we also need to define suitable operations over these truth values, and we want these operations to provide a natural interpretation for the logical connectives. Which operations are appropriate?

Well, the terms we are working with are just propositional formulas. So how would we go about evaluating a formula χ in the truth value algebra? Obviously we would have to know whether the proposition letters in χ are true or false, but let's suppose that this has been taken care of by a function $\theta : \Phi \to 2$ mapping the set Φ of proposition letters to the set 2 of truth values. Given such a θ (logicians will call θ a valuation, algebraists will call it an assignment) it is clear what we have to do: compute $\tilde{\theta}(\phi)$ according to the following rules:

$$\begin{array}{rcl} \tilde{\theta}(p) & = & \theta(p), \text{ for all } p \in \Phi \\ \tilde{\theta}(\bot) & = & 0 \\ \tilde{\theta}(\neg\phi) & = & 1 - \tilde{\theta}(\phi) \\ \tilde{\theta}(\phi \vee \psi) & = & \max(\tilde{\theta}(\phi), \tilde{\theta}(\psi)). \end{array} \tag{5.1}$$

Clearly the operations used here are the relevant ones; they simply restate the familiar truth table definitions. This motivates the following definition:

Definition 5.2 The *algebra of truth values* is $\mathbf{2} = (\{0, 1\}, +, -, 0))$, where $-$ and $+$ are defined by $-a = 1 - a$ and $a + b = \max(a, b)$, respectively. $\dashv$

Let's sum up our discussion so far. The crucial observations are that formulas can be viewed as terms, that valuations can be identified with algebraic assignments in the algebra **2**, and that evaluating the truth of a formula under such a valuation/assignment is exactly the same as determining the meaning of the term in the algebra **2** under the assignment/valuation.

So let's move on. We have viewed meaning as a map $\tilde{\theta}$ from the set $Form(\Phi)$ to the set $\{0, 1\}$ — but it is useful to consider this *meaning function* in more mathematical detail. Note the 'shape' of the conditions on $\tilde{\theta}$ in (5.1): the resemblance to the defining condition of a *homomorphism* is too blatant to miss. But since homomorphisms are the fundamental maps between algebras (see Appendix B) why not try and impose algebraic structure on the *domain* of such meaning functions (that is, on the set of formulas/terms) so that meaning functions really are homomorphisms? This is exactly what we are about to do. We first define the needed algebraic structure on the set of formulas.

Definition 5.3 Let Φ a set of propositional variables. The propositional *formula algebra* over Φ is the algebra

$$\mathfrak{Form}(\Phi) = (Form(\Phi), +, -, \bot),$$

where Φ is the collection of propositional formulas over Φ, and $-$ and $+$ are the operations defined by $-\phi := \neg\phi$ and $\phi + \psi := \phi \vee \psi$, respectively. $\dashv$

In other words, the carrier of this algebra is the collection of propositional formulas over the set of proposition letters Φ, and the operations $-$ and $+$ give us a simple mathematical picture of the dynamics of formula construction.

Proposition 5.4 *Let Φ be some set of proposition letters. Given any assignment $\theta : \Phi \to 2$, the function $\tilde{\theta} : Form(\Phi) \to 2$ assigning to each formula its meaning under this valuation, is a homomorphism from $\mathfrak{Form}(\Phi)$ to* **2**.

Proof. A precise definition of homomorphism is given in Appendix B. Essentially, homomorphisms between algebras map elements in the source algebra to elements in the target algebra in an operation preserving way — and this is precisely what the conditions in $\tilde{\theta}$ in (5.1) express. $\dashv$

The idea of viewing formulas as terms, and meaning as a homomorphism, is fundamental to algebraic logic.

Another point is worth stressing. As the reader will have noticed, sometimes we call a sequence of symbols like $p \vee q$ a formula, and sometimes we call it a term. This is intentional. Any propositional formula can be viewed as — simply *is* — an algebraic term. The one-to-one correspondence involved is so obvious that it's not worth talking about 'translating' formulas to terms or vice-versa; they're simply two ways of looking at the same thing. We simply choose whichever terminology seems most appropriate to the issue under discussion.

But let's move on. As is clear from high-school algebra, algebraic reasoning is essentially *equational*. So a genuinely *algebraic* logic of propositions should give us a way of determining when two propositions are equal. For example, such a logic should be capable of determining that the formulas $p \vee (q \wedge p)$ and p denote the same proposition. How does the algebraic approach to propositional semantics handle this? As follows: an equation $s \approx t$ is valid in an algebra $\mathfrak{A}$ if for every assignment to the variables occurring in the terms, s and t have the same meaning in $\mathfrak{A}$ (see Appendix B for further details). Hence, an algebraic way of saying that a formula ϕ is a classical tautology (notation: $\models_C \phi$) is to say that the equation $\phi \approx \top$ is valid in the algebra of truth values.

Now, an attractive feature of propositional logic (a feature which extends to modal logic) is that not only terms, but *equations* correspond to formulas. There is nothing mysterious about this: we can define the bi-implication connective $\leftrightarrow$ in classical propositional logic, and viewed as an operation on propositions, $\leftrightarrow$ asserts that both terms have the same meaning:

$$\tilde{\theta}(\phi \leftrightarrow \psi) \;=\; \begin{cases} 1 & \text{if } \tilde{\theta}(\phi) = \tilde{\theta}(\psi) \\ 0 & \text{otherwise.} \end{cases}$$

So to speak, propositional logic is intrinsically equational.

Theorem 5.5 neatly summarizes our discussion so far: it shows how easily we can move from a logical to an algebraic perspective and back again.

Theorem 5.5 (2 Algebraizes Classical Validity) *Let ϕ and ψ be propositional formulas/terms. Then*

$$\models_C \phi \quad \textit{iff} \quad \mathbf{2} \models \phi \approx \top \tag{5.2}$$

$$\mathbf{2} \models \phi \approx \psi \quad \textit{iff} \quad \models_C \phi \leftrightarrow \psi. \tag{5.3}$$

$$\models_C \phi \leftrightarrow (\phi \leftrightarrow \top) \tag{5.4}$$

Proof. Immediate from the definitions. ⊣

Remark 5.6 The reader may wonder about the presence of (5.3) and in particular, of (5.4) in the Theorem. The point is that for a proper, 'full', algebraization of a logic, one has to establish not only that the membership of some formula ϕ in the logic can be rendered algebraically as the validity of some equation $\phi^{\approx}$ in some (class of) algebra(s). One also has to show that conversely, there is a translation of equations to formulas such that the equation holds in the class of algebras if and only if its translation belongs to the logic. And finally, one has to prove that translating a formula ϕ to an equation $\phi^{\approx}$, and then translating this equation back to a formula, one obtains a formula ϕ' that is *equivalent* to the original formula ϕ. The fact that our particular translations satisfy these requirements is stated by (5.3) and (5.4), respectively.

Since we will not go far enough into the theory of algebraic logic to use these 'full' algebraizations, in the sequel we will only mention the first kind of equivalence when we algebraize a logic. Nevertheless, in all the cases that we consider, the second and third requirements are met as well. $\dashv$

Set algebras

Propositional formulas/terms and equations may be interpreted in any algebra of type $Bool$. Most algebra of this type are uninteresting as far as the semantics of propositional logic is concerned — but other algebras besides **2** are relevant. A particularly important example is the class of *set algebras*. As we will now see, set algebras provide us with a second algebraic perspective on the semantics of propositional logic. And as we will see in the following section, the perspective they provide extends neatly to modal logic.

Definition 5.7 (Set Algebras) Let A be a set. As usual, we denote the *power set* of A (the set of all subsets of A) by $\mathcal{P}(A)$. The *power set algebra* $\mathfrak{P}(A)$ is the structure

$$\mathfrak{P}(A) = (\mathcal{P}(A), \cup, -, \varnothing),$$

where $\varnothing$ denotes the *empty* set, $-$ is the operation of taking the *complement* of a set relative to A, and $\cup$ that of taking the *union* of two sets. From these basic operations we define in the standard way the operation $\cap$ of taking the *intersection* of two sets, and the special element A, the *top set* of the algebra.

A *set algebra* or *field of sets* is a subalgebra of a power set algebra. That is, a set algebra (on A) is a collection of subsets of A that contains $\varnothing$ and is closed under $\cup$ and $-$ (so any set algebra contains A and is closed under $\cap$ as well). The class of all set algebras is called Set. $\dashv$

Set algebras provide us with a simple concrete picture of propositions and the way they are combined — moreover, it's a picture that even at this stage contains a number of traditional modal ideas. Think of A as a set of worlds (or situations, or states) and think of a proposition as a subset of A. And think of a proposition as a set of worlds — the worlds that make it true. So viewed, $\varnothing$ is a very special proposition: it's the proposition that is false in every situation, which is clearly a good way of thinking about the meaning of $\bot$. Similarly, A is the proposition true in all situations, which is a suitable meaning for $\top$. It should also be clear that $\cup$ is a way of combining propositions that mirrors the role of $\vee$. After all, in what worlds is $p \vee q$ true? In precisely those worlds that make either p true or q true. Finally, complementation mirrors negation, for $\neg p$ is true in precisely those worlds where p is not true.

As we will now show, set algebras and the algebra **2** make precisely the same equations true. We'll prove this algebraically by showing that the class of set algebras coincides (modulo isomorphism) to the class of subalgebras of powers of **2**. The crucial result needed is the following:

Proposition 5.8 *Every power set algebra is isomorphic to a power of* **2***, and conversely.*

Proof. Let A be an arbitrary set, and consider the following function χ mapping elements of $\mathcal{P}(A)$ to 2-valued maps on A:

$$\chi(X)(a) = \begin{cases} 1 & \text{if } a \in X \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $\chi(X)$ is the *characteristic function* of X. The reader should verify that χ is an isomorphism between $\mathfrak{P}(A)$ and $\mathbf{2}^A$.

Conversely, to show that every power of **2** is isomorphic to some power set algebra, let $\mathbf{2}^I$ be some power of **2**. Consider the map $\alpha : 2^I \to \mathcal{P}(I)$ defined by

$$\alpha(f) = \{i \in I \mid f(i) = 1\}.$$

Again, we leave it for the reader to verify that α is the required isomorphism between $\mathbf{2}^I$ and $\mathfrak{P}(I)$. ⊣

Theorem 5.9 (Set **algebraizes classical validity)** *Let ϕ and ψ be propositional formulas/terms. Then*

$$\models_C \phi \quad \textit{iff} \quad \mathsf{Set} \models \phi \approx \top. \tag{5.5}$$

Proof. It is not difficult to show from first principles that the validity of equations is preserved under taking direct products (and hence powers) and subalgebras. Thus, with the aid of Theorem 5.5 and Proposition 5.8, the result follows. ⊣

Algebraizing propositional axiomatics

We now have two equational perspectives on the semantics of propositional logic: one via the algebra **2**, the other via set algebras. But what about the syntactic aspects of propositional logic? It's time to see how the equational perspective handles such notions as theoremhood and provable equivalence.

Assume we are working in some fixed (sound and complete) proof system for classical propositional logic. Let $\vdash_C \phi$ mean that ϕ is a theorem of this system, and call two propositional formulas ϕ and ψ *provably equivalent* (notation: $\psi \equiv_C \psi$)

if the formula $\phi \leftrightarrow \psi$ is a theorem. Theorem 5.11 below is a syntactic analog of Theorem 5.9: it is the fundamental result concerning the algebraization of propositional axiomatics. Its statement and proof makes use of *boolean algebras*, so lets define these important entities right away.

Definition 5.10 (Boolean Algebras) Let $\mathfrak{A} = (A, +, -, 0)$ be an algebra of the boolean similarity type. Then $\mathfrak{A}$ is called a *boolean algebra* iff it satisfies the following identities:

$(B0)$	$x + y = y + x$	$x \cdot y = y \cdot x$
$(B1)$	$x + (y + z) = (x + y) + z$	$x \cdot (y \cdot z) = (x \cdot y) \cdot z$
$(B2)$	$x + 0 = x$	$x \cdot 1 = 1$
$(B3)$	$x + (-x) = 1$	$x \cdot (-x) = 0$
$(B4)$	$x + (y \cdot z) = (x + y) \cdot (x + z)$	$x \cdot (y + z) = (x \cdot y) + (x \cdot z)$

We *order* the elements of a boolean algebra by defining $a \leq b$ if $a + b = b$ (or equivalently, if $a \cdot b = a$). Given a boolean algebra $\mathfrak{A} = (A, +, -, 0)$, the set A is called its *carrier set*. We call the class of boolean algebras BA. ⊣

By a famous result of Birkhoff's (discussed in Appendix B) a class of algebras defined by a collection of equations is what is known as a *variety*. Thus in what follows we sometimes speak of the variety of boolean algebras, rather than the class of boolean algebras.

If you haven't encountered boolean algebras before, you should check that the algebra **2** and the set algebras defined earlier are both examples of boolean algebras (that is, check that these algebras satisfy the listed identities). In fact, set algebras are what are known as *concrete* boolean algebras. As we will see when we discuss the Stone Representation Theorem, the relationship between abstract boolean algebras (that is, any algebraic structure satisfying the previous definition) and set algebras lies at the heart of the algebraic perspective on propositional soundness and completeness.

But this is jumping ahead: our immediate task is to state the syntactic analog of Theorem 5.9 promised above.

Theorem 5.11 (BA Algebraizes Classical Theoremhood) *Let ϕ and ψ be propositional formulas/terms. Then*

$$\vdash_C \phi \quad \textit{iff} \quad \mathsf{BA} \models \phi \approx \top. \tag{5.6}$$

Proof. Soundness (the direction from left to right in (5.6) can be proved by a straightforward inductive argument on the length of propositional proofs. Completeness will follow from the Propositions 5.14 and 5.15 below. ⊣

How are we to prove this completeness result? Obviously we have to show that every non-theorem of classical propositional logic can be falsified on some boolean algebra (falsified in the sense that there is some assignment under which the formula does not evaluate to the top element of the algebra). So the key question is: how do we build falsifying algebras? Our earlier work on relational completeness suggests an answer. In Chapter 4 we made use of canonical models: that is, we manufactured models out of syntactical ingredients (sets of formulas) taking care to hardwire in all the crucial facts about the logic. So the obvious question is: can we construct algebras from (sets of) formulas in a way that builds in all the propositional logic we require? Yes, we can. Such algebras are called Lindenbaum-Tarski algebras. In essence, they are 'canonical algebras'.

First some preliminary work. The observation underpinning what follows is that the relation of provable equivalence is a *congruence* on the formula algebra. A congruence on an algebra is essentially an equivalence relation on the algebra that respects the operations (a precise definition is given in Appendix B) and it is not hard to see that provable equivalence is such a relation.

Proposition 5.12 *The relation $\equiv_C$ is a congruence on the propositional formula algebra.*

Proof. We have to prove that $\equiv_C$ is an equivalence relation satisfying

$$\phi \equiv_C \psi \text{ only if } \neg\phi \equiv_C \neg\psi \tag{5.7}$$

and

$$\phi_0 \equiv_C \psi_0 \text{ and } \phi_1 \equiv_C \psi_1 \text{ only if } (\phi_0 \vee \phi_1) \equiv_C (\psi_0 \vee \psi_1). \tag{5.8}$$

In order to prove that $\equiv_C$ is reflexive, we have to show that for any formula ϕ, the formula $\phi \leftrightarrow \phi$ is a theorem of the proof system. The reader is invited to prove this in his or her favorite proof system for proposition calculus. The properties of symmetry and transitivity are also left to the reader.

But we want to prove that $\equiv_C$ is not merely an equivalence relation but a congruence. We deal with the case for negation, leaving (5.8) to the reader. Suppose that $\phi \equiv_C \psi$, that is, $\vdash_C \phi \leftrightarrow \psi$. Again, given that we are working with a sound and complete proof system for propositional calculus, this implies that $\vdash_C \neg\phi \leftrightarrow \neg\psi$. Given this, (5.7) is immediate. ⊣

The equivalence classes under $\equiv_C$ are the building blocks for what follows. As any such class is a maximal set of mutually equivalent formulas, we can think of such classes as propositions. And as $\equiv_C$ is a congruence, we can define a natural algebraic structure on these propositions. Doing so gives rise to Lindenbaum-Tarski algebras.

Definition 5.13 (Lindenbaum-Tarski Algebra) Given a set of proposition letters Φ, let $Form(\Phi)/{\equiv_C}$ be the set of equivalence classes that $\equiv_C$ induces on the set of formulas, and for any formula ϕ let $[\phi]$ denote the equivalence class containing ϕ. Then the *Lindenbaum-Tarski algebra* (for this language) is the structure

$$\mathfrak{L}_C(\Phi) := (Form(\Phi)/{\equiv_C}, +, -, 0),$$

where +, $-$ and 0 are defined by: $[\phi] + [\psi] := [\phi \vee \psi]$, $-[\phi] := [\neg\phi]$ and $0 := [\bot]$. Strictly speaking, we should write $[\phi]_\Phi$ instead of $[\phi]$, for ϕ's congruence class depends on the set Φ of proposition letters. But unless there is potential for confusion, we usually won't bother to do so. $\dashv$

Lindenbaum-Tarski algebras are easy to work with. As an example, we show that $a + (-a) = 1$ for all elements a of $\mathfrak{L}_C(\Phi)$. The first observation is that a, just like *any* element of $\mathfrak{L}_C(\Phi)$, is of the form $[\phi]$ for some formula ϕ. But then we have

$$a + (-a) = [\phi] + (-[\phi]) = [\phi] + [\neg\phi] = [\phi \vee (\neg\phi)] = [\top] = 1, \qquad (5.9)$$

where the fourth equality holds because $\vdash_C (\phi \vee \neg\phi) \leftrightarrow \top$.

We need two results concerning Lindenbaum-Tarski algebras. First, we have to show that they are indeed an 'algebraic canonical model' — that is, that they give us a counterexample for *every* non-theorem of propositional logic. Second, we have to show that they are counterexamples of the right kind: that is, we need to prove that any Lindenbaum-Tarski algebra is a boolean algebra.

Proposition 5.14 *Let ϕ be some propositional formula, and Φ a set of proposition letters of size not smaller than the number of proposition letters occurring in ϕ. Then*

$$\vdash_C \phi \textit{ iff } \mathfrak{L}_C(\Phi) \models \phi \approx \top. \qquad (5.10)$$

Proof. We may and will assume that Φ actually contains all variables occurring in ϕ, cf. Exercise 5.1.4. We first prove the easy direction from right to left. Assume that ϕ is *not* a theorem of classical propositional logic. This implies that ϕ and $\top$ are not provably equivalent, whence we have $[\phi] \neq [\top]$. We have to find an assignment on $\mathfrak{L}_C(\Phi)$ that forms a counterexample to the validity of ϕ. There is one obvious candidate, namely the assignment ι given by $\iota(p) = [p]$. It can easily be verified (by a straightforward formula induction) that with this definition we obtain $\tilde{\iota}(\psi) = [\psi]$ for *all* formulas ψ that use variables from the set Φ. But then by our assumption on ϕ we find that

$$\tilde{\iota}(\phi) = [\phi] \neq [\top] = 1,$$

as required.

For the other direction we have to work a bit harder. If $\vdash_C \phi$ then it is obvious that $\tilde{\iota}(\phi) = [\phi] = [\top] = 1$, but only looking at ι is not sufficient now. We have to show that $\tilde{\theta}(\phi) = [\top]$ for *all* assignments θ.

So let θ be an arbitrary assignment. That is, θ assigns an equivalence class (under $\equiv_C$) to each propositional variable. For each variable p, take a representing formula $\rho(p)$ in the equivalence class $\theta(p)$; that is, we have $\theta(p) = [\rho(p)]$. We may view ρ as a *function* mapping propositional variables to formulas; in other words, ρ is a *substitution*. Let $\rho(\psi)$ denote the effect of performing this substitution on the formula ψ. It can be proved by an easy formula induction that, for any formula ψ, we have

$$\tilde{\theta}(\psi) = [\rho(\psi)]. \tag{5.11}$$

Now, the collection of propositional theorems is closed under uniform substitution (depending on the formulation of your favorite sound and complete proof system, this is either something that is hardwired in or can be shown to hold). This closure property implies that the formula $\rho(\phi)$ is a theorem, and hence that $\rho(\phi) \equiv_C \top$, or equivalently, $[\rho(\phi)] = [\top]$. But then it follows from (5.11) that

$$\tilde{\theta}(\phi) = [\top],$$

which is precisely what we need to show that $\mathfrak{L}_C(\Phi) \models \phi$. ⊣

Thus it only remains to check that $\mathfrak{L}_C(\Phi)$ is the right kind of algebra.

Proposition 5.15 *For any set Φ of proposition letters, $\mathfrak{L}_C(\Phi)$ is a boolean algebra.*

Proof. Fix a set Φ. The proof of this Proposition boils down to proving that all the identities B0–4 hold in $\mathfrak{L}_C(\Phi)$. In (5.9) above we proved that the first part of B3 holds; we leave the reader to verify that the other identities hold as well. ⊣

Summarizing, we have seen that the axiomatics of propositional logic can be algebraized in a class of algebras, namely the variety of boolean algebras. We have also seen that Lindenbaum-Tarski algebras act as canonical representatives of the class of boolean algebras. (For readers with some background in universal algebra, we remark that Lindenbaum-Tarski algebras are in fact the *free* boolean algebras.)

Weak completeness via Stone

It's time to put our findings together, and to take one final step. This step is more important than any taken so far.

Theorem 5.9 captured tautologies as equations valid in set algebras:

$$\models_C \phi \text{ iff } \mathsf{Set} \models \phi \approx \top.$$

On the other hand, in Theorem 5.11 we found an algebraic semantics for the notion of classical theoremhood:

$$\vdash_C \phi \text{ iff } \mathsf{BA} \models \phi \approx \top,$$

But there is a fundamental *logical* connection between $\models_C$ and $\vdash_C$: the soundness and completeness theorem for propositional logic tells us that they are identical. Does this crucial connection show up algebraically? That is, is there an algebraic analog of the soundness and completeness result for classical propositional logic? There is: it's called the Stone Representation Theorem.

Theorem 5.16 (Stone Representation Theorem) *Any boolean algebra is isomorphic to a set algebra.*

Proof. We will make a more detailed statement of this result, and prove it, in Section 5.3. ⊣

(Incidentally, this immediately tells us that any boolean algebra is isomorphic to a subalgebra of a power of **2** — for Proposition 5.8 tells us that any power set algebra is isomorphic to a subalgebra of a power of **2**.) But what really interests us here is the *logical* content of Stone's Theorem. In essence, it is the key to the weak completeness of classical propositional logic.

Corollary 5.17 (Soundness and Weak Completeness) *For any formula ϕ, ϕ is valid iff it is a theorem.*

Proof. Immediate from the equations above, since by the Stone Representation Theorem, the equations valid in Set must coincide with those valid in BA. ⊣

The relation between Theorem 5.11 and Corollary 5.17 is the key to much of our later work. Note that from a logical perspective, Corollary 5.17 is the interesting result: it establishes the soundness and completeness of classical propositional logic with respect to the standard semantics. So why is Theorem 5.11 important? After all, as it proves completeness with respect to an abstractly defined class of boolean algebras, it doesn't have the same independent logical interest. This is true, but given that the abstract algebraic counterexamples it provides can be represented as standard counterexamples — and this is precisely what Stone's theorem guarantees — it enables us to prove the standard completeness result for propositional logic.

To put it another way, the algebraic approach to completeness factors the algebra building process into two steps. We first prove completeness with respect to an abstract algebraic semantics by building an abstract algebraic model. It's easy to do this — we just use Lindenbaum-Tarski algebras. We then try and *represent* the abstract algebras in the concrete form required by the standard semantics.

In the next two sections we extend this approach to modal logic. Algebraizing modal logic is more demanding than algebraizing propositional logic. For a start, there isn't just one logic to deal with — we want to be able to handle any normal modal logic whatsoever. Moreover, the standard semantics for modal logic is given in terms of frame-based models — so we are going to need a representation result that tells us how to represent algebras as relational structures.

But all this can be done. In the following section we'll generalize boolean algebras to boolean algebras with operators; these are the *abstract* algebras we will be dealing with throughout the chapter. We also generalize set algebras to complex algebras; these are the *concrete* algebras which model the idea of set-based algebras of propositions for modal languages. We then define the Lindenbaum-Tarski algebras we need — and every normal modal logic will give rise to its own Lindenbaum-Tarski algebra. This is all a fairly straightforward extension of ideas we have just discussed. We then turn, in Section 5.3, to the crucial representation result: the Jónsson-Tarski Theorem. This is an extension of Stone's Representation Theorem that tells us how to represent a boolean algebra with operators as an ordinary modal model. It is an elegant result in its own right, but for our purposes its importance is the bridge it provides between completeness in the universe of algebras and completeness in the universe of relational structures.

Exercises for Section 5.1

5.1.1 Let A and B be two sets, and $f : A \to B$ some map. Show that $f^{-1} : \mathcal{P}(B) \to \mathcal{P}(A)$ given by $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$ is a *homomorphism* from the power set algebra of B to that of A.

5.1.2 Prove that every power set algebra is isomorphic to a power of the algebra $\mathbf{2}$, and that conversely, every power of $\mathbf{2}$ is isomorphic to a power set algebra.

5.1.3 Here's a standard set of axioms for propositional calculus: $p \to (q \to p)$, $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$, and $(\neg p \to \neg q) \to (q \to p)$. Show that all three axioms are valid on any set algebra. That is, show that whatever subset is used to interpret the propositional variables, these formulas are true in all worlds. Furthermore, show that modus ponens and uniform substitution preserve validity.

5.1.4 Let Φ and Ψ be two sets of proposition letters.

(a) Prove that $\mathfrak{Form}(\Phi)$ is a subalgebra of $\mathfrak{Form}(\Psi)$ iff $\Phi \subseteq \Psi$.
(b) Prove that $\mathfrak{L}_C(\Phi)$ can be *embedded* in $\mathfrak{L}_C(\Psi)$ iff $|\Phi| \leq |\Psi|$.
(c) Prove that $\mathfrak{L}_C(\Phi)$ and $\mathfrak{L}_C(\Psi)$ are isomorphic iff $|\Phi| = |\Psi|$.
(d) Does $\Phi \subseteq \Psi$ imply that $\mathfrak{L}_C(\Phi)$ is a subalgebra of $\mathfrak{L}_C(\Psi)$?

5.2 Algebraizing Modal Logic

Let's adapt the ideas introduced in the previous section to modal logic. The most basic principle of algebraic logic is that formulas of a logical language can be viewed as terms of an algebraic language, so let's first get clear about the algebraic languages we will use in the remainder of this chapter:

Definition 5.18 Let τ be a modal similarity type. The *corresponding algebraic similarity type* $\mathcal{F}_\tau$ contains as function symbols all modal operators, together with the boolean symbols $\vee$ (binary), $\neg$ (unary), and $\bot$ (constant). For a set Φ of variables, we let $Ter_\tau(\Phi)$ denote the collection of $\mathcal{F}_\tau$-terms over Φ. $\dashv$

The algebraic similarity type $\mathcal{F}_\tau$ can be seen as the union of the modal similarity type τ and the boolean type $Bool$. In practice we often identify τ and $\mathcal{F}_\tau$, speaking of τ-terms instead of $\mathcal{F}_\tau$-terms. The previous definition takes the formulas-as-terms paradigm quite literally: by our definitions

$$Form(\tau, \Phi) = Ter_\tau(\Phi).$$

Just as boolean algebras were the key to the algebraization of classical propositional logic, in modal logic we are interested in *boolean algebras with operators* or BAOs. Let's first define BAOs abstractly; we'll discuss concrete BAOs shortly.

Definition 5.19 (Boolean Algebras with Operators) Let τ be a modal similarity type. A *boolean algebra with τ-operators* is an algebra

$$\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle \in \tau}$$

such that $(A, +, -, 0)$ is a boolean algebra and every $f_\triangle$ is an *operator* of arity $\rho(\triangle)$; that is, $f_\triangle$ is an operation satisfying

(Normality) $f_\triangle(a_1, \ldots, a_{\rho(\triangle)}) = 0$ whenever $a_i = 0$ for some i ($0 < i \leq \rho(\triangle)$).
(Additivity) For all i (such that $0 < i \leq \rho(\triangle)$),

$$f_\triangle(a_1, \ldots, a_i + a_i', \ldots, a_{\rho(\triangle)}) = f_\triangle(a_1, \ldots, a_i, \ldots, a_{\rho(\triangle)}) + f_\triangle(a_1, \ldots, a_i', \ldots, a_{\rho(\triangle)}).$$

If we abstract from the particular modal similarity type τ, or if τ is known from context, we simply speak of *boolean algebras with operators*, or BAO*s*. $\dashv$

Now, the boolean structure is obviously there to handle the propositional connectives, but what is the meaning of the normality and additivity conditions on the $f_\triangle$? Consider a unary operator f. In this case these conditions boil down to:

$$\begin{aligned} f(0) &= 0 \\ f(x+y) &= fx + fy. \end{aligned}$$

But these equations correspond to the following modal formulas:

$$\begin{aligned} \Diamond\bot &\leftrightarrow \bot \\ \Diamond(p \vee q) &\leftrightarrow \Diamond p \vee \Diamond q, \end{aligned}$$

both of which formulas are modal validities. Indeed (as we noted in Remark 4.7) they can be even be used to axiomatize the minimal normal logic **K**. Thus, even at this stage, it should be clear that our algebraic *operators* are well named: their defining properties are modally crucial.

Furthermore, note that all operators have the property of *monotonicity*. An operation g on a boolean algebra is *monotonic* if $a \leq b$ implies $ga \leq gb$. (Here $\leq$ refers to the ordering on boolean algebra given in Definition 5.10: $a \leq b$ iff $a \cdot b = a$ iff $a + b = b$.) Operators are monotonic, because if $a \leq b$, then $a + b = b$, so $fa + fb = f(a + b) = fb$, and so $fa \leq fb$. Once again there is an obvious modal analog, namely the rule of proof mentioned in Remark 4.7: if $\vdash_\Lambda p \to q$ then $\vdash_\Lambda \Diamond p \to \Diamond q$.

Example 5.20 Consider the collection of binary relations over a given set U. This collection forms a set algebra on which we can define the operations $\mid$ (composition), $(\cdot)^{-1}$ (inverse) and *Id* (the identity relation); these are binary, unary and nullary operations respectively. It is easy to verify that these operations are actually operators; to give a taste of the kind of argumentation required, we show that composition is additive in its second argument:

$(x, y) \in R \mid (S \cup T)$
iff there is a z with $(x, z) \in R$ and $(z, y) \in S \cup T$
iff there is a z with $(x, z) \in R$ and $(z, y) \in S$ or $(z, y) \in T$
iff there is a z with $(x, z) \in R$ and $(z, y) \in S$,
or there is a z with $(x, z) \in R$ and $(z, y) \in T$
iff $(x, y) \in R \mid S$ or $(x, y) \in R \mid T$
iff $(x, y) \in R \mid S \cup R \mid T$.

The reader should check the remaining cases. ⊣

Algebraizing modal semantics

However it is the next type of BAO that is destined to play the starring role:

Definition 5.21 (Complex Algebras) Given an $(n + 1)$-ary relation R on a set W, we define the n-ary operation m_R on subsets of W by

$$m_R(X_1, \ldots, X_n) = \{w \in W \mid Rww_1 \ldots w_n \text{ for some } w_1 \in X_1, \ldots, w_n \in X_n\}$$

Now let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\triangle)_{\triangle \in \tau}$ a τ-frame. The *(full) complex algebra of* $\mathfrak{F}$ (notation: $\mathfrak{F}^+$), is the extension of the power set algebra $\mathfrak{P}(W)$ with operations $m_{R_\triangle}$ for every operator $\triangle$ in τ. A *complex algebra* is a subalgebra of a full complex algebra. If K is a class of frames, then we denote the class of full complex algebras of frames in K by $\mathbf{Cm}\mathsf{K}$. ⊣

It is important that you fully understand this definition. For a start, note that complex algebras are set algebras (that is, concrete propositional algebras) to which

m_R operations have been added. What are these m_R? Actually, we've met them before: they were defined in Definition 1.30. There we mentioned that for a *binary* relation R, the unary operation m_R yields the set of all states which 'see' a state in a given subset X of the universe:

$$m_R(X) = \{y \in W \mid \text{ there is an } x \in X \text{ such that } Ryx\}.$$

Given an n-tuple of subsets of the universe, the n-ary operation m_R returns the set of all states which 'see' an n-tuple of states each of which belongs to the corresponding subset. It easily follows that if we have some model in mind and denote with $\tilde{V}(\phi)$ the set of states where ϕ is true, then

$$m_{R_\triangle}(\tilde{V}(\phi_1, \ldots, \phi_n)) \;=\; \tilde{V}(\triangle(\phi_1, \ldots, \phi_n)).$$

Thus it should be clear that complex algebras are intrinsically modal. In the previous section we said that set algebras model propositions as sets of possible worlds. By adding the m_R, we've modeled the idea that one world may be able to access the information in another. In short, we've defined a class of concrete algebras which capture the modal notion of access between states in a natural way.

How are complex algebras connected with abstract BAOs? One link is obvious:

Proposition 5.22 *Let τ be a modal similarity type, and $\mathfrak{F} = (W, R_\triangle)_{\triangle\in\tau}$ a τ-frame. Then $\mathfrak{F}^+$ is a boolean algebra with τ-operators.*

Proof. We have to show that operations of the form m_R are normal and additive. This rather easy proof is left to the reader; see Exercise 5.2.2. ⊣

The other link is deeper. As we will learn in the following section (Theorem 5.43), complex algebras are to BAOs what set algebras are to boolean algebras: every abstract boolean algebra with operators has a concrete set theoretic representation, for every boolean algebra with operators is isomorphic to a complex algebra.

But we have a lot to do before we are ready to prove this — let us continue our algebraization of the semantics of modal logic. We will now define the interpretation of τ-terms and equations in arbitrary boolean algebras with τ-operators. As we saw for propositional logic, the basic idea is very simple: given an assignment that tells us what the variables stand for, we can inductively define the meaning of any term.

Definition 5.23 Assume that τ is a modal similarity type and that Φ is a set of variables. Assume further that $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle\in\tau}$ is a boolean algebra with τ-operators. An *assignment* for Φ is a function $\theta : \Phi \to A$. We can extend θ uniquely to a meaning function $\tilde{\theta}$: $Ter_\tau(\Phi) \to A$ satisfying:

$$\tilde{\theta}(p) \;=\; \theta(p), \text{ for all } p \in \Phi$$

$$\begin{array}{rcl}\tilde{\theta}(\bot) & = & 0 \\ \tilde{\theta}(\neg s) & = & -\tilde{\theta}(s) \\ \tilde{\theta}(s \vee t) & = & \tilde{\theta}(s) + \tilde{\theta}(t) \\ \tilde{\theta}(\triangle(s_1, \ldots, s_n)) & = & f_\triangle(\tilde{\theta}(s_1), \ldots, \tilde{\theta}(s_n)).\end{array}$$

Now let $s \approx t$ be a τ-equation. We say that $s \approx t$ is *true* in $\mathfrak{A}$ (notation: $\mathfrak{A} \models s \approx t$) if for every assignment θ: $\tilde{\theta}(s) = \tilde{\theta}(t)$. ⊣

But now consider what happens when $\mathfrak{A}$ is a *complex* algebra $\mathfrak{F}^+$. Since elements of $\mathfrak{F}^+$ are *subsets* of the power set $\mathcal{P}(W)$ of the universe W of $\mathfrak{F}$, assignments θ are simply ordinary modal valuations! The ramifications of this observation are listed in the following proposition:

Proposition 5.24 *Let τ be a modal similarity type, ϕ a τ-formula, $\mathfrak{F}$ a τ-frame, θ an assignment (or valuation) and w a point in $\mathfrak{F}$. Then*

$$(\mathfrak{F}, \theta), w \Vdash \phi \quad \textit{iff} \quad w \in \tilde{\theta}(\phi), \tag{5.12}$$

$$\mathfrak{F} \Vdash \phi \quad \textit{iff} \quad \mathfrak{F}^+ \models \phi \approx \top, \tag{5.13}$$

$$\mathfrak{F}^+ \models \phi \approx \psi \quad \textit{iff} \quad \mathfrak{F} \Vdash \phi \leftrightarrow \psi. \tag{5.14}$$

Proof. We will only prove the first part of the proposition (for the basic modal similarity type); the second and third part follow immediately from this and the definitions.

Let ϕ, $\mathfrak{F}$ and θ be as in the statement of the theorem. We will prove (5.12) (for all w) by induction on the complexity of ϕ. The only interesting part is the modal case of the inductive step. Assume that ψ is of the form $\Diamond\chi$. The key observation is that

$$\tilde{\theta}(\Diamond\phi) = m_{R_\Diamond}(\tilde{\theta}(\phi)). \tag{5.15}$$

We now have:

$$\begin{array}{rll}(\mathfrak{F}, \theta), w \Vdash \Diamond\chi & \text{iff} & \text{there is a } v \text{ such that } R_\Diamond wv \text{ and } (\mathfrak{F}, \theta), v \Vdash \chi \\ & \text{iff} & \text{there is a } v \text{ such that } R_\Diamond wv \text{ and } v \in \tilde{\theta}(\chi) \\ & \text{iff} & w \in m_R(\tilde{\theta}(\chi)) \\ & \text{iff} & w \in \tilde{\theta}(\Diamond\chi).\end{array}$$

Here the second equivalence is by the inductive hypothesis, and the last one by (5.15). This proves (5.12). ⊣

The previous proposition is easily lifted to the level of classes of frames and complex algebras. The resulting theorem is a fundamental one: it tells us that classes of complex algebras algebraize modal semantics. It is the modal analog of Theorem 5.9.

Theorem 5.25 *Let τ be a modal similarity type, ϕ and ψ τ-formulas, $\mathfrak{F}$ a τ-frame, θ an assignment (or valuation) and w a point in $\mathfrak{F}$. Then*

$$\mathsf{K} \Vdash \phi \quad \textit{iff} \quad \mathbf{Cm}\mathsf{K} \models \phi \approx \top, \tag{5.16}$$

$$\mathbf{Cm}\mathsf{K} \models \phi \approx \psi \quad \textit{iff} \quad \mathsf{K} \Vdash \phi \leftrightarrow \psi. \tag{5.17}$$

Proof. Immediate by Proposition 5.24. ⊣

This proposition allows us to identify the *modal logic* Λ_{K} of a class of frames K (that is, the set of formulas that are valid in each $\mathfrak{F} \in \mathsf{K}\}$) with the *equational theory* of the class $\mathbf{Cm}\mathsf{K}$ of complex algebras of frames in K (that is, the set of equations $\{s \approx t \mid \mathfrak{F}^+ \models s \approx t$, for all $\mathfrak{F} \in \mathsf{K}\}$).

Let us summarize what we learned so far. We have developed an algebraic approach to the semantics of modal logic in terms of complex algebras. These complex algebras, *concrete* boolean algebras with operators, generalize to modal languages the idea of algebras of propositions provided by set algebras. And most important of all, we have learned that complex algebras embody *all* the information about normal modal logics that frames do. Thus, mathematically speaking, we can dispense with frames and instead work with complex algebras.

Algebraizing modal axiomatics

Turning to the algebraization of modal axiomatics, we encounter a situation similar to that of the previous section. Once again, we will see that the algebraic counterpart of a logic is an equational class of algebras. To give a precise formulation we need the following definition.

Definition 5.26 Given a formula ϕ, let $\phi^{\approx}$ be the equation $\phi \approx \top$. Now let τ be a modal similarity type. For a set Σ of τ-formulas, we define V_{Σ} to be the class of those boolean algebras with τ-operators in which the set $\Sigma^{\approx} = \{\sigma^{\approx} \mid \sigma \in \Sigma\}$ is valid. ⊣

We now state the algebraic completeness proof we wish to prove. It is the modal analog of Theorem 5.11.

Theorem 5.27 (Algebraic Completeness) *Let τ be a modal similarity type, and Σ a set of τ-formulas. Then $\mathbf{K}_{\tau}\Sigma$ (the normal modal τ-logic axiomatized by Σ) is sound and complete with respect to V_{Σ}. That is, for all formulas ϕ we have*

$$\vdash_{\mathbf{K}_{\tau}\Sigma} \phi \ \textit{ iff } \ \mathsf{V}_{\Sigma} \models \phi^{\approx}.$$

Proof. We leave the soundness direction as an exercise to the reader. Completeness is an immediate corollary of Theorems 5.32 and 5.33 below. ⊣

As a corollary to the soundness direction of Theorem 5.27, we have that $\mathsf{V}_{\mathbf{K}_\tau\Sigma} = \mathsf{V}_\Sigma$, for any set Σ of formulas. In the sequel this will allow us to forget about axiom sets and work with logics instead.

To prove the completeness direction of Theorem 5.27, we need a modal version of the basic tool used to prove algebraic completeness results: Lindenbaum-Tarski algebras. As in the the case of propositional languages, we will build an algebra on top of the set of formulas in such a way that the relation of provable equivalence between two formulas is a congruence relation. The key difference is that we don't have just one relation of provable equivalence, but many: we want to define the notion of Lindenbaum-Tarski algebras for arbitrary normal modal logics.

Definition 5.28 Let τ be an algebraic similarity type, and Φ a set of propositional variables. The formula algebra of τ over Φ is the algebra $\mathfrak{Form}(\tau, \Phi) = (Form(\tau, \Phi), +, -, \bot, f_\triangle)_{\triangle \in \tau}$ where $+$, $-$ and $\bot$ are given as in Definition 5.3, while for each modal operator, the operation $f_\triangle$ is given by

$$f_\triangle(t_1, \dots, t_n) = \triangle(t_1, \dots, t_n). \quad \dashv$$

Notice the double role of $\triangle$ in this definition: on the right hand side of the equation, $\triangle$ is a 'static' part of the *term* $\triangle(t_1, \dots, t_n)$, whereas in the left hand side we have a more 'dynamic' perspective on the *interpretation* $f_\triangle$ of the operation symbol $\triangle$.

Definition 5.29 Let τ be a modal similarity type, Φ a set of propositional variables, and Λ a normal modal τ-logic. We define $\equiv_\Lambda$ as a binary relation between τ-formulas (in Φ) by

$$\phi \equiv_\Lambda \psi \text{ iff } \vdash_\Lambda \phi \leftrightarrow \psi.$$

If $\phi \equiv_\Lambda \psi$, we say that ϕ and ψ are *equivalent modulo* Λ. $\dashv$

Proposition 5.30 *Let τ be a modal similarity type, Φ a set of proposition letters and Λ a normal modal τ-logic. Then $\equiv_\Lambda$ is a congruence relation on $\mathfrak{Form}(\tau, \Phi)$.*

Proof. We confine ourselves to proving the proposition for the basic modal similarity type. First, we have to show that $\equiv_\Lambda$ is an equivalence relation; this is easy, and we leave the details to the reader. Next, we must show that $\equiv_\Lambda$ is a *congruence* relation on the formula algebra; that is, we have to demonstrate that $\equiv_\Lambda$ has the following properties:

$$\begin{array}{rll} \phi_0 \equiv_\Lambda \psi_0 \text{ and } \phi_1 \equiv_\Lambda \psi_1 & \text{imply} & \phi_0 \vee \phi_1 \equiv_\Lambda \phi_0 \vee \psi_1 \\ \phi \equiv_\Lambda \psi & \text{implies} & \neg\phi \equiv_\Lambda \neg\psi \\ \phi \equiv_\Lambda \psi & \text{implies} & \Diamond\phi \equiv_\Lambda \Diamond\psi. \end{array} \tag{5.18}$$

The first two properties are easy exercises in propositional logic. The third is an immediate corollary of Lemma 4.6. $\dashv$

Proposition 5.30 tells us that the following are correct definitions of functions on the set $Form(\tau, \Phi)/{\equiv_\Lambda}$ of equivalence classes under $\equiv_\Lambda$:

$$\begin{array}{rcl} [\phi] + [\psi] & := & [\phi \vee \psi] \\ -[\phi] & := & [\neg\phi] \\ f_\triangle([\phi_1], \ldots, [\phi_n]) & := & [\triangle(\phi_1, \ldots, \phi_n)] \end{array} \tag{5.19}$$

For unary diamonds, the last clause boils down to: $f_\Diamond[\phi] := [\Diamond\phi]$.

Given Proposition 5.30, the way is open to define the Lindenbaum-Tarski algebra for any normal modal logic Λ: we simply define it to be the *quotient algebra* of the formula algebra over the congruence relation $\equiv_\Lambda$.

Definition 5.31 (Lindenbaum-Tarski Algebras) Let τ be a modal similarity type, Φ a set of propositional variables, and Λ a normal modal τ-logic in this language. The *Lindenbaum-Tarski algebra of Λ over the set of generators Φ* is the structure

$$\mathfrak{L}_\Lambda(\Phi) := (Form(\tau, \Phi)/{\equiv_\Lambda}, +, -, f_\triangle),$$

where the operations $+$, $-$, and $f_\triangle$ are defined as in (5.19). $\dashv$

As with propositional logic, we need two results about Lindenbaum-Tarski algebras. First, we must show that modal Lindenbaum-Tarski algebras are boolean algebras with operators; indeed, we need to show that the Lindenbaum-Tarski algebra of any normal modal logic Λ belongs to V_Λ. Second, we need to prove that Lindenbaum-Tarski algebras provide canonical counterexamples to the validity of non-theorems of Λ in V_Λ. The second point is easily dealt with:

Theorem 5.32 *Let τ be a modal similarity type, and Λ a normal modal τ-logic. Let ϕ be some propositional formula, and Φ a set of proposition letters of size not smaller than the number of proposition letters occurring in ϕ. Then*

$$\vdash_\Lambda \phi \text{ iff } \mathfrak{L}_\Lambda(\Phi) \models \phi^\approx. \tag{5.20}$$

Proof. This proof is completely analogous to that of Proposition 5.14 and is left to the reader. $\dashv$

So let's verify that Lindenbaum-Tarski algebras are canonical algebraic models of the right kind:

Theorem 5.33 *Let τ be a modal similarity type, and Λ be a normal modal τ-logic. Then for any set Φ of proposition letters, $\mathfrak{L}_\Lambda(\Phi)$ belongs to V_Λ.*

Proof. Once we have shown that $\mathfrak{L}_\Lambda(\Phi)$ is a boolean algebra with τ-operators, the theorem immediately follows from Theorem 5.32. Now, that $\mathfrak{L}_\Lambda(\Phi)$ is a boolean algebra is clear, so the only thing that remains to be done is to show that the modalities really give rise to τ-operators.

As an example, assume that τ contains a diamond $\Diamond$; let us prove additivity of $f_\Diamond$. We have to show that

$$f_\Diamond(a+b) = f_\Diamond a + f_\Diamond b,$$

for arbitrary elements a and b of $\mathfrak{L}_\Lambda(\Phi)$. Let a and b be such elements; by definition there are formulas ϕ and ψ such that $a = [\phi]$ and $b = [\psi]$. Then

$$f_\Diamond(a+b) = f_\Diamond([\phi]+[\psi]) = f_\Diamond([\phi \vee \psi]) = [\Diamond(\phi \vee \psi)]$$

while

$$f_\Diamond a + f_\Diamond b = f_\Diamond([\phi]) + f_\Diamond([\psi]) = [\Diamond\phi] + [\Diamond\psi] = [\Diamond\phi \vee \Diamond\psi].$$

It is easy to check that

$$\vdash_\Lambda \Diamond(\phi \vee \psi) \leftrightarrow (\Diamond\phi \vee \Diamond\psi),$$

whence it follows that $[\Diamond(\phi \vee \psi)] = [\Diamond\phi \vee \Diamond\psi]$. We leave it for the reader to fill in the remaining details of this proof as Exercise 5.2.4. $\dashv$

As an immediate corollary we have the following result: modal logics are always complete with respect to the variety of boolean algebras with operators where their axioms are valid. This is in sharp contrast to the situation in relational semantics, where (as we saw in Chapter 4) modal logics need *not* be complete with respect to the class of frames that they define.

This is an interesting result, but it's not what we really want, for it proves completeness with respect to abstract BAOs rather than complex algebras. Not only are complex algebras concrete algebras of propositions, we also know (recall Proposition 5.24) that complex algebras embody all the information of relevance to frame validity — so we really should be aiming for completeness results with respect to classes of complex algebras.

And that's why the long-promised Jónsson-Tarski theorem, which we state and prove in the following section, is so important. This tells us that *every* boolean algebra with operators is isomorphic to a complex algebra, and thus guarantees that we can represent the Lindenbaum-Tarski algebras of any normal modal logics Λ as a complex algebra. In effect, it will convert Theorem 5.32 into a completeness result with respect to complex algebras. Moreover, because of the link between complex algebras and relational semantics, it will open the door to exploring frame completeness algebraically.

Exercises for Section 5.2

5.2.1 Let $\mathfrak{A}$ be a boolean algebra. Prove that $\cdot$ is an operator. How about $+$?

5.2.2 Show that every complex algebra is a boolean algebra with operators (that is, prove Proposition 5.22).

5.2.3 Let A be the collection of finite and co-finite subsets of $\mathbb{N}$. Define $f : A \to A$ by

$$f(X) = \begin{cases} \{y \in \mathbb{N} \mid y + 1 \in X\} & \text{if } X \text{ is finite,} \\ \mathbb{N} & \text{if } X \text{ is cofinite.} \end{cases}$$

Prove that $(A, \cup, -, \varnothing, f)$ is a boolean algebra with operators.

5.2.4 Let Λ be a normal modal logic. Prove that the Lindenbaum-Tarski algebra $\mathfrak{L}_\Lambda$ is a boolean algebra with τ-operators (that is, fill in the missing proof details in Theorem 5.33).

5.2.5 Let Σ be a set of τ-formulas. Prove that for any formula ϕ, $\vdash_{\mathbf{K}_\tau \Sigma} \phi$ implies $sfV_\Sigma \models \phi^{\approx}$. That is, prove the soundness direction of Theorem 5.27.

5.2.6 Call a variety V of BAOs *complete* if it is generated by a class of complex algebras, i.e., if $\mathsf{V} = \mathbf{HSPCm}\mathsf{K}$ for some frame class K. Prove that a logic Λ is complete iff the variety V_Λ is complete.

5.2.7 Let $\mathfrak{A}$ be a boolean algebra. In this exercise we assume familiarity with the notion of an infinite sum (supremum). An operation $f : A \to A$ is called *completely additive* if it distributes over infinite sums (in each of its arguments).

(a) Show that every operation of the form m_R is completely additive.
(b) Give an example of an operation that is additive, but not completely additive. (Hint: as the boolean algebra, take the set of finite and co-finite subsets of some frame.)

5.3 The Jónsson-Tarski Theorem

We already know how to construct a BAO from a frame: simply form the frame's complex algebra. We will now learn how to construct a frame from a BAO by forming the *ultrafilter frame* of the algebra. As we will see, this operation generalizes two constructions that we have met before: taking the ultrafilter extension of a model, and forming the canonical frame associated with a normal modal logic.

Our new construction will lead us to the desired representation theorem: by taking the complex algebra of the ultrafilter frame of a BAO, we obtain the *canonical embedding algebra* of the original BAO. The fundamental result of this section (and indeed, of the entire chapter) is that *every boolean algebra with operators can be isomorphically embedded in its canonical embedding algebra*. We will prove this result and along the way discuss a number of other important issues, such as the algebraic status of canonical models and ultrafilter extensions, and the importance of canonical varieties of BAOs for modal completeness theory.

Let us consider the problem of (isomorphically) embedding an arbitrary BAO $\mathfrak{A}$ in a complex algebra. Obviously, the first question to ask is: what should the underlying *frame* of the complex algebra be? To keep our notation simple, let us assume for the moment that we are working in a similarity type with just one unary modality, and that $\mathfrak{A} = (A, +, -, 0, f)$ is a boolean algebra with one unary operator f. Thus we have to find a universe W and a binary relation R on W such that $\mathfrak{A}$ can be embedded in the complex algebra of the frame (W, R). Stone's Representation Theorem 5.16 gives us half the answer, for it tells us how to embed the boolean part of $\mathfrak{A}$ in the power set algebra of the set $Uf\mathfrak{A}$ of *ultrafilters* of $\mathfrak{A}$. Let's take a closer look at this fundamental result.

Stone's representation theorem

We have already met filters and ultrafilters in Chapter 2, when we defined the ultrafilter extension of a model. Now we generalize these notions to the context of *abstract* boolean algebras.

Definition 5.34 A *filter* of a boolean algebra $\mathfrak{A} = (A, +, -, 0)$ is a subset $F \subseteq A$ satisfying

(F1) $1 \in F$
(F2) F is closed under taking meets; that is, if $a, b \in F$ then $a \cdot b \in F$
(F3) F is upward closed; that is, if $a \in F$ and $a \leq b$ then $b \in F$.

A filter is *proper* if it does not contain the smallest element 0, or, equivalently, if $F \neq A$. An *ultrafilter* is a proper filter satisfying

(F4) For every $a \in A$, either a or $-a$ belongs to F.

The collection of ultrafilters of $\mathfrak{A}$ is called $Uf\mathfrak{A}$. $\dashv$

Note the difference in terminology: an (ultra)filter *over* the set W is an (ultra)filter *of* the power set algebra $\mathfrak{P}(W)$.

Example 5.35 For any element a of a boolean algebra $\mathfrak{B}$, the set $a\uparrow = \{b \in a \mid a \leq b\}$ is a filter. In the field of finite and co-finite subsets of a countable set A, the collection of co-finite subsets of A forms an *ultrafilter*. $\dashv$

Example 5.36 Since the collection of filters of a boolean algebra is closed under taking intersections, we may speak of the *smallest* filter F_D containing a given set $D \subseteq A$. This filter can also be defined as the following set:

$$\{a \in A \mid \text{ there are } d_0, \ldots, d_n \in D \text{ such that } d_1 \cdot \ldots \cdot d_n \leq a\} \tag{5.21}$$

which explains why we will also refer to F_D as the filter *generated by* D. This filter is proper if D has the so-called *finite meet property*; that is, if there is no finite subset $\{d_0, \ldots, d_n\}$ of D such that $d_1 \cdot \ldots \cdot d_n = 0$. $\dashv$

For future reference, we gather some properties of ultrafilters; the proof of the next proposition is left to the reader.

Proposition 5.37 *Let* $\mathfrak{A} = (A, +, -, 0)$ *be a boolean algebra. Then*

(i) *For any ultrafilter* u *of* $\mathfrak{A}$ *and for every pair of elements* $a, b \in A$ *we have that* $a + b \in u$ *iff* $a \in u$ *or* $b \in u$.
(ii) *Uf*$\mathfrak{A}$ *coincides with the set of maximal proper filters on* $\mathfrak{A}$ *('maximal' is understood with respect to set inclusion).*

The main result that we need in the proof of Stone's Theorem is the Ultrafilter Theorem: this guarantees that there are enough ultrafilters for our purposes.

Proposition 5.38 (Ultrafilter Theorem) *Let* $\mathfrak{A}$ *be a boolean algebra,* a *an element of* A*, and* F *a proper filter of* $\mathfrak{A}$ *that does not contain* a*. Then there is an ultrafilter extending* F *that does not contain* a.

Proof. We first prove that every proper filter can be extended to an ultrafilter. Let G be a proper filter of $\mathfrak{A}$, and consider the set X of all proper filters H extending G. Suppose that Y is a *chain* in X; that is, Y is a nonempty subset of X of which the elements are pairwise ordered by set inclusion. We leave it to the reader to verify that $\bigcup Y$ is a proper filter; obviously, $\bigcup Y$ extends G; so $\bigcup Y$ belongs to X itself. This shows that X is closed under taking unions of chains, whence it follows from Zorn's Lemma that X contains a maximal element u. We claim that u is an ultrafilter.

For suppose otherwise. Then there is a $b \in A$ such that neither b nor $-b$ belongs to u. Consider the filters H and H' generated by $u \cup \{b\}$ and $u \cup \{-b\}$, respectively. Since neither of these can belong to X, both must be improper; that is, $0 \in H$ and $0 \in H'$. But then by definition there are elements $u_1, \ldots, u_n, u'_1, \ldots, u'_m$ in u such that

$$u_1 \cdot \ldots \cdot u_n \cdot b \leq 0 \text{ and } u'_1 \cdot \ldots \cdot u'_m \cdot -b \leq 0.$$

From this it easily follows that

$$u_1 \cdot \ldots \cdot u_n \cdot u'_1 \cdot \ldots \cdot u'_m = 0,$$

contradicting the fact that u is a proper filter.

Now suppose that a and F are as in the statement of the proposition. It is not hard to show that $F \cup \{-a\}$ is a set with the finite meet property. In Example 5.36 we saw that there is a proper filter G extending F and containing $-a$. Now we use the first part of the proof to find an ultrafilter u extending G. But if u extends G it also extends F, and if it contains $-a$ it cannot contain a. $\dashv$

It follows from Proposition 5.38 and the facts mentioned in Example 5.36 that any subset of a boolean algebra can be extended to an ultrafilter provided that it has the finite meet property. We now have all the necessary material to prove Stone's Theorem.

Theorem 5.16 (Stone Representation Theorem) *Any boolean algebra is isomorphic to a field of sets, and hence, to a subalgebra of a power of* $\mathbf{2}$*. As a consequence, the variety of boolean algebras is generated by the algebra* $\mathbf{2}$*:*

$$\mathsf{BA} \;=\; \mathbb{V}(\{\mathbf{2}\}).$$

Proof. Fix a boolean algebra $\mathfrak{A} = (A, +, -, 0)$. We will embed $\mathfrak{A}$ in the power set of $Uf\mathfrak{A}$. Consider the map $\rho : A \to \mathcal{P}(Uf\mathfrak{A})$ defined as follows:

$$\rho(a) \;=\; \{u \in Uf\mathfrak{A} \mid a \in u\}.$$

We first show that ρ is a homomorphism. As an example we treat the join operation:

$$\begin{aligned} \rho(a+b) &= \{u \in Uf\mathfrak{A} \mid a+b \in u\} \\ &= \{u \in Uf\mathfrak{A} \mid a \in u \text{ or } b \in u\} \\ &= \{u \in Uf\mathfrak{A} \mid a \in u\} \cup \{u \in Uf\mathfrak{A} \mid b \in u\} \\ &= \rho(a) \cup \rho(b). \end{aligned}$$

Note that the crucial second equality follows from Proposition 5.37.

It remains to prove that ρ is injective. Suppose that a and b are distinct elements of A. We may derive from this that either $a \not\leq b$ or $b \not\leq a$. Without loss of generality we may assume the second. But if $b \not\leq a$ then a does not belong to the filter $b{\uparrow}$ generated by $\{b\}$, so by Proposition 5.38 there is some ultrafilter u such that $b{\uparrow} \subseteq u$ and $a \notin u$. Obviously, $b{\uparrow} \subseteq u$ implies that $b \in u$. But then we have that $u \in \rho(b)$ and $u \notin \rho(a)$

This shows that $\mathfrak{A}$ is isomorphic to a field of sets; it then follows by Proposition 5.8 that $\mathfrak{A}$ is isomorphic to a subalgebra of a power of $\mathbf{2}$. From this it is immediate that BA is the variety generated by the algebra $\mathbf{2}$. ⊣

Remark 5.39 That every boolean algebra is isomorphic to a subalgebra of a power of the algebra $\mathbf{2}$ can be proved more directly by observing that there is a 1–1 correspondence between ultrafilters of $\mathfrak{A}$ and homomorphisms from $\mathfrak{A}$ onto $\mathbf{2}$. Given an ultrafilter u of $\mathfrak{A}$, define $\alpha_u : \mathfrak{A} \to \mathbf{2}$ by

$$\alpha_u(a) = \begin{cases} 1 & \text{if } a \in u \\ 0 & \text{otherwise} \end{cases}$$

And conversely, given a homomorphism $\alpha : \mathfrak{A} \to \mathbf{2}$, define the ultrafilter u_α by

$$u_\alpha \;=\; \alpha^{-1}(1) \;\; (= \{a \in \mathfrak{A} \mid \alpha(a) = 1\}).$$

We leave further details to the reader. ⊣

Ultrafilter frames

Now that we have a candidate for the universe of the ultrafilter frame of a given BAO $\mathfrak{A}$, let us see how to define a relation R on ultrafilters such that we can embed $\mathfrak{A}$ in the algebra $(Uf\mathfrak{A}, R)^+$. To motivate the definition of R, we will view the elements of the algebra as *propositions*, and imagine that $r(a)$ (the representation map r applied to proposition a) yields the set of states where a is *true* according to some valuation. Hence, reading fa as $\Diamond a$, it seems natural that a state u should be in $r(fa)$ if and only if there is a v with Ruv and $v \in r(a)$. So, in order to decide whether Ruv should hold for two arbitrary states (ultrafilters) u and v, we should look at all the propositions a holding at v (that is, all elements $a \in v$) and check whether fa holds at u (that is, whether $fa \in u$). Putting it more formally, the natural, 'canonical' choice for R seems to be the relation Q_f given by

$$Q_f uv \text{ iff } fa \in u \text{ for all } a \in v.$$

The reader should compare this definition with the definition of the canonical relation given in Definition 4.18. Although one is couched in terms of ultrafilters, and the other in terms of maximal consistent sets (MCSs), both clearly trade on the same idea. As we will shortly learn (and as the above identification of 'ultrafilters' and 'maximal sets of propositions' already suggests), this is no accident.

In the general case, we use the following definition (an obvious analog of Definition 4.24).

Definition 5.40 Let $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle\in\tau}$ be a boolean algebra with operators. The $(n+1)$-ary relation Q_f on the set $Uf\mathfrak{A}$ of ultrafilters of $\mathfrak{A}$ is given by

$$Q_f uu_1 \ldots u_n \text{ iff } f(a_1, \ldots, a_n) \in u \text{ for all } a_1 \in u_1, \ldots, a_n \in u_n$$

The frame $(Uf\mathfrak{A}, Q_{f_\triangle})_{\triangle\in\tau}$ is called the *ultrafilter frame* of $\mathfrak{A}$ (notation: $\mathfrak{A}_+$). The complex algebra $(\mathfrak{A}_+)^+$ is called the *(canonical) embedding algebra* of $\mathfrak{A}$ (notation: $\mathfrak{Em}\mathfrak{A}$). $\dashv$

We leave it to the reader to verify that the *ultrafilter extension* $\mathfrak{ue}\,\mathfrak{F}$ of a frame $\mathfrak{F}$ is nothing but the ultrafilter frame of the complex algebra of $\mathfrak{F}$, in symbols: $\mathfrak{ue}\,\mathfrak{F} = (\mathfrak{F}^+)_+$.

For later reference, we state the following proposition (an obvious analog of Lemma 4.25) which shows that we could have given an alternative but equivalent definition of the relation Q_f.

Proposition 5.41 *Let f be an n-ary operator on the boolean algebra $\mathfrak{A}$, and u, $u_1, \ldots, u_n$ an $(n+1)$-tuple of ultrafilters of $\mathfrak{A}$. Then*

$$Q_f uu_1 \ldots u_n \textit{ iff } -f(-a_1, \ldots, -a_n) \in u \textit{ implies that for some } i,\ a_i \in u_i.$$

Proof. We only prove the direction from left to right. Suppose that $Q_f uu_1 \dots u_n$, and that $-f(-a_1, \dots, -a_n) \in u$. To arrive at a contradiction, suppose that there is no i such that $a_i \in u_i$. But as $Q_f uu_1 \dots u_n$, it follows that $f(-a_1, \dots, -a_n) \in u$. But this contradicts the fact that $-f(-a_1, \dots, -a_n) \in u$. $\dashv$

As the above sequence of analogous definitions and results suggest, we have already encountered a kind of frame which is very much like an ultrafilter frame, namely the *canonical frame* of a normal modal logic (see Definition 4.18). The basic idea should be clear now: the states of the canonical frame are the MCSs of the logic, and an ultrafilter is nothing but an abstract version of an MCS. But this is no mere analogy: the canonical frame of a logic is actually *isomorphic* to the ultrafilter frame of its Lindenbaum-Tarski algebra, and the mapping involved is simple and intuitive. When making this connection, the reader should keep in mind that when we defined 'the' canonical frame in Chapter 4, we always had a fixed, countable set Φ of proposition letters in mind.

Theorem 5.42 *Let τ be a modal similarity type, Λ a normal modal τ-logic, and Φ the set of propositional variables used to define the canonical frame $\mathfrak{F}^\Lambda$. Then*

$$\mathfrak{F}^\Lambda \cong (\mathfrak{L}_\Lambda(\Phi))_+.$$

Proof. We leave it to the reader to show that the function θ defined by

$$\theta(\Gamma) = \{[\phi] \mid \phi \in \Gamma\},$$

mapping a maximal Λ-consistent set Γ to the set of equivalence classes of its members, is the required isomorphism between $\mathfrak{F}^\Lambda$ and $(\mathfrak{L}_\Lambda(\Phi))_+$. $\dashv$

The Jónsson-Tarski Theorem

We are ready to prove the Jónsson-Tarski Theorem: every boolean algebra with operators is embeddable in the full complex algebra of its ultrafilter frame.

Theorem 5.43 (Jónsson-Tarski Theorem) *Let τ be a modal similarity type, and $\mathfrak{A} = (A, +, -, 0, f_\triangle)_{\triangle \in \tau}$ be a boolean algebra with τ-operators. Then the representation function $r : A \to \mathcal{P}(Uf\mathfrak{A})$ given by*

$$r(a) = \{u \in Uf\mathfrak{A} \mid a \in u\}$$

is an embedding of $\mathfrak{A}$ into $\mathfrak{Em}\mathfrak{A}$.

Proof. To simplify our notation a bit, we work in a similarity type with a single n-ary modal operator, assuming that $\mathfrak{A} = (A, +, -, 0, f)$ is a boolean algebra with a single n-ary operator f. By Stone's Representation Theorem, the map $r : A \to \mathcal{P}(Uf(\mathfrak{A}))$ given by

$$r(x) = \{u \in Uf(A) \mid x \in u\}$$

is a boolean embedding. So, it suffices to show that r is also a *modal* homomorphism; that is, that

$$r(f(a_1, \ldots, a_n)) = m_{Q_f}(r(a_1), \ldots, r(a_n)). \tag{5.22}$$

We will first prove (5.22) for unary f. In other words, we have to prove that

$$r(fa) = m_{Q_f}(r(a)).$$

We start with the inclusion from right to left: assume $u \in m_{Q_f}(r(a))$. Then by definition of m_{Q_f}, there is an ultrafilter u_1 with $u_1 \in r(a)$ (that is, $a \in u_1$) and $Q_f uu_1$. By definition of Q_f this implies $fa \in u$, or $u \in r(fa)$.

For the other inclusion, let u be an ultrafilter in $r(fa)$, that is, $fa \in u$. To prove that $u \in m_{Q_f}(r(a))$, it suffices to find an ultrafilter u_1 such that $Q_f uu_1$ and $u_1 \in r(a)$, or $a \in u_1$. The basic idea of the proof is that we first pick out those elements of A (other than a) that we cannot avoid putting in u_1. These elements are given by the condition $Q_f uu_1$. By Proposition 5.41 we have that for every element of the form $-f(-y)$ in u, y has to be in u_1; therefore, we define

$$F := \{y \in A \mid -f(-y) \in u\}.$$

We will now show that there is an ultrafilter $u_1 \supseteq F$ containing a. First, an easy proof (using the additivity of f), shows that F is closed under taking meets. Second, we prove that

$$F' := \{a \cdot y \mid y \in F\}$$

has the finite meet property. As F is closed under taking meets, it is sufficient to show that $a \cdot y \neq 0$ whenever $y \in F$. To arrive at a contradiction, suppose that $a \cdot y = 0$. Then $a \leq -y$, so by the monotonicity of f, $fa \leq f(-y)$; therefore, $f(-y) \in u$, contradicting $y \in F$.

By Theorem 5.38 there is an ultrafilter $u_1 \supseteq F'$. Note that $a \in u_1$, as $1 \in F$. Finally, $Q_f uu_1$ holds by definition of F: if $-f(-y) \in u$ then $y \in F \subseteq u_1$.

We now prove (5.22) for arbitrary $n \geq 1$ by induction on the arity n of f. We have just proved the base case. So, assume that the induction hypothesis holds for n. We only treat the direction from left to right, since the other direction can be proved as in the base case. Let f be a normal and additive function of rank $n + 1$, and suppose that $a_1, \ldots, a_{n+1}$ are elements of $\mathfrak{A}$ such that $f(a_1, \ldots, a_{n+1}) \in u$. We have to find ultrafilters $u_1, \ldots, u_{n+1}$ of $\mathfrak{A}$ such that (i) $a_i \in u_i$ for all i with $1 \leq i \leq n + 1$, and (ii) $Q_f uu_1 \ldots u_{n+1}$. Our strategy will be to let the induction hypothesis take care of $u_1, \ldots, u_n$ and then to search for u_{n+1}.

Let $f' : A^n \to A$ be the function given by

$$f'(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, a_{n+1}).$$

That is, for the time being we fix a_{n+1}. It is easy to see that f' is normal and additive, so we may apply the induction hypothesis. Since $f'(a_1, \ldots, a_n) \in u$, this yields ultrafilters $u_1, \ldots, u_n$ such that $a_i \in u_i$ for all i with $1 \leq i \leq n$, and

$$f(x_1, \ldots, x_n, a_{n+1}) \in u, \text{ whenever } x_i \in u_i \ (1 \leq i \leq n). \tag{5.23}$$

Now we'll define an ultrafilter u_{n+1} such that $a_{n+1} \in u_{n+1}$ and $Q_f uu_1 \ldots u_{n+1}$. This second condition can be rewritten as follows (we abbreviate '$x_1 \in u_1, \ldots, x_n \in u_n$' by '$\vec{x} \in \vec{u}$'):

$Q_f uu_1 \ldots u_{n+1}$
iff for all $\vec{x}, y$: if $\vec{x} \in \vec{u}$, then $y \in u_{n+1}$ implies $f(\vec{x}, y) \in u$
iff for all $\vec{x}, y$: if $\vec{x} \in \vec{u}$, then $f(\vec{x}, y) \notin u$ implies $y \notin u_{n+1}$
iff for all $\vec{x}, y$: if $\vec{x} \in \vec{u}$, then $-f(\vec{x}, y) \in u$ implies $-y \in u_{n+1}$
iff for all $\vec{x}, z$: if $\vec{x} \in \vec{u}$, then $-f(\vec{x}, -z) \in u$ implies $z \in u_{n+1}$.

This provides us with a minimal set of elements that u_{n+1} should contain; put

$$F := \{z \in A \mid \exists \vec{x} \in \vec{u}\, (-f(\vec{x}, -z) \in u)\}.$$

If $-f(\vec{x}, -z) \in u$, we say that $\vec{x}$ *drives* z *into* F. We now take the first condition into account as well, defining $F' := \{a_{n+1}\} \cup F$.

Our aim is to prove the existence of an ultrafilter u_{n+1} containing F'. It will be clear that this is sufficient to prove the theorem (note that $a_{n+1} \in F'$ as $1 \in F$). To be able to apply the Ultrafilter Theorem 5.38, we will show that F' has the finite meet property. We first need the following fact:

$$F \text{ is closed under taking meets.} \tag{5.24}$$

Let z', z'' be in F; assume that z' and z'' are driven into F by $\vec{x}'$ and $\vec{x}''$, respectively. We will now see that $\vec{x} := (x_1' \cdot x_1'', \ldots, x_n' \cdot x_n'')$ drives $z := z' \cdot z''$ into F, that is, that $-f(\vec{x}, -z) \in u$.

Since f is monotonic, we have $f(\vec{x}, -z') \leq f(\vec{x}', -z')$, and hence we find that $-f(\vec{x}', -z') \leq -f(\vec{x}, -z')$. As u is upward closed and $-f(\vec{x}', -z') \in u$ by our 'driving assumption,' this gives $-f(\vec{x}, -z') \in u$. In the same way we find $-f(\vec{x}, -z'') \in u$. Now

$$f(\vec{x}, -z) = f(\vec{x}, -(z' \cdot z'')) = f(\vec{x}, (-z') + (-z'')) = f(\vec{x}, -z') + f(\vec{x}, -z''),$$

whence

$$-f(\vec{x}, -z) = [-f(\vec{x}, -z')] \cdot [-f(\vec{x}, -z'')].$$

Therefore, $-f(\vec{x}, -z) \in u$, since u is closed under taking meets. This proves (5.24).

We can now finish the proof and show that indeed

$$F' \text{ has the finite meet property.} \tag{5.25}$$

By (5.24) it suffices to show that $a_{n+1} \cdot z \neq 0$ for all $z \in F$. To prove this, we reason by contraposition: suppose that $z \in F$ and $a_{n+1} \cdot z = 0$. Let $\vec{x} \in \vec{u}$ be a sequence that drives z into F, that is, $-f(\vec{x}, -z) \in u$. From $a_{n+1} \cdot z = 0$ it follows that $a_{n+1} \leq -z$, so by monotonicity of f we get $-f(\vec{x}, -z) \leq -f(\vec{x}, a_{n+1})$. But then $-f(\vec{x}, a_{n+1}) \in u$, which contradicts (5.23). This proves that indeed $a_{n+1} \cdot z \neq 0$ and hence we have shown (5.25) and thus, Theorem 5.43. ⊣

Canonicity

To conclude this section, let's discuss the significance of this result. Clearly the Jónsson-Tarski Theorem guarantees that we can represent the Lindenbaum-Tarski algebras of normal modal logics as complex algebras, so it immediately converts Theorem 5.32 into a completeness result with respect to complex algebras.

But we want more: because of the link between complex algebras and relational semantics, it seems to offer a plausible algebraic handle on frame completeness. And in fact it does — but we need to be careful. As should be clear from our work in Chapter 4, even with the Jónsson-Tarski Theorem at our disposal, one more hurdle remains to be cleared. In Exercise 5.2.6 we defined the notion of a *complete* variety of BAOs: a variety V is complete if there is a frame class K that generates V in the sense that $\mathsf{V} = \mathbf{HSPCm}\mathsf{K}$. The exercise asked the reader to show that any logic Λ is complete if and only if V_Λ is a complete variety. Now does the Jónsson-Tarski Theorem establish such a thing? Not really — it *does* show that every algebra $\mathfrak{A}$ is a complex algebra over *some* frame, thus proving that for any logic Λ we have that $\mathsf{V}_\Lambda \subseteq \mathbf{Cm}\mathsf{K}$ for some frame class K. So, this certainly gives $\mathsf{V} \subseteq \mathbf{HSPCm}\mathsf{K}$. However, in order to prove completeness, we have to establish an equality instead of an inclusion. One way to prove this is to show that the complex algebras that we have found form a subclass of V_Λ. That is, show that for any algebra $\mathfrak{A}$ in the variety V_Λ, the frame $\mathfrak{A}_+$ is a frame for the logic Λ. This requirement gives us an algebraic handle on the notion of canonicity.

Let's examine a concrete example. Recall that $\mathbf{K4}$ is the normal logic generated by the 4 axiom, $\Diamond\Diamond p \to \Diamond p$. We know from Theorem 4.27 that $\mathbf{K4}$ is complete with respect to the class of transitive frames. How can we prove this result algebraically?

A little thought reveals that the following is required: we have to show that the Lindenbaum-Tarski algebras for $\mathbf{K4}$ are embeddable in full complex algebras of *transitive* frames. Recall from Section 3.3 that the 4 axiom *characterizes* the transitive frames, thus in our proposed completeness proof, we would have to show that 4 is valid in the ultrafilter frame $(\mathfrak{L}_{\mathbf{K4}}(\Phi))_+$ of $\mathfrak{L}_{\mathbf{K4}}(\Phi)$, or equivalently, that $((\mathfrak{L}_{\mathbf{K4}}(\Phi))_+)^+$ belongs to the variety V_4. Note that by Theorem 5.32 we already know that $\mathfrak{L}_{\mathbf{K4}}(\Phi)$ belongs to V_4.

As this example suggests, proving frame completeness results for extensions of **K** algebraically leads directly to the following question: *which varieties of* BAO*s are closed under taking canonical embedding algebras?* In fact, this is the required algebraic handle on canonicity and motivates the following definition.

Definition 5.44 Let τ be a modal similarity type, and C a class of boolean algebras with τ-operators. C is *canonical* if it is closed under taking canonical embedding algebras; that is, if for all algebras $\mathfrak{A}$, $\mathfrak{Em}\mathfrak{A}$ is in C whenever $\mathfrak{A}$ is in C. ⊣

Thus we now have two notions of canonicity, namely the logical one of Definition 4.30 and the algebraic one just defined. Using Theorem 5.32, we show that these two concepts are closely related.

Proposition 5.45 *Let τ be a modal similarity type, and Σ a set of τ-formulas. If V_Σ is a canonical variety, then Σ is canonical.*

Proof. Assume that the variety V_Σ is canonical, and let Φ be the fixed countable set of proposition letters that we use to define canonical frames. By Theorem 5.32, the Lindenbaum-Tarski algebra $\mathfrak{L}_{\mathbf{K\Sigma}}(\Phi)$ is in V_Σ; then, by assumption, its canonical embedding algebra $\mathfrak{Em}\mathfrak{L}_{\mathbf{K\Sigma}}$ is in V_Σ. However, from Theorem 5.42 it follows that this algebra is isomorphic to the complex algebra of the canonical frame of $\mathbf{K\Sigma}$:

$$\mathfrak{Em}\mathfrak{L}_{\mathbf{K\Sigma}}(\Phi) = ((\mathfrak{L}_{\mathbf{K\Sigma}}(\Phi))_+)^+ \cong (\mathfrak{F}^{\mathbf{K\Sigma}})^+.$$

Now the fact that $(\mathfrak{F}^{\mathbf{K\Sigma}})^+$ is in V_Σ means that $\mathfrak{F}^{\mathbf{K\Sigma}} \Vdash \Sigma$ by Proposition 5.24. But this implies that Σ is canonical. ⊣

An obvious question is whether the converse of Proposition 5.45 holds as well; that is, whether a variety V_Σ is canonical if Σ is a canonical set of modal formulas. However, note that canonicity of Σ only implies that one *particular* boolean algebra with operators has its embedding algebra in V_Σ, namely the Lindenbaum-Tarski algebra over a *countably* infinite number of generators. In fact, we are facing an open problem here:

Open Problem 1 *Let τ be a modal similarity type, and Σ a canonical set of τ-formulas. Is V_Σ a canonical variety?*

Equivalently, suppose that E is a set of equations such that for all countable *boolean algebras with τ-operators we have the following implication*

$$\text{if } \mathfrak{A} \models E \text{ then } \mathfrak{Em}\mathfrak{A} \models E. \tag{5.26}$$

Is V_E a canonical variety? In other words, does (5.26) hold for all *boolean algebras with τ-operators?*

In spite of this unresolved issue, the algebraic notion of canonicity can do a lot of work for us. The important point is that it offers a genuinely new perspective on what canonicity is, a perspective that will allow us use algebraic arguments. This will be demonstrated in Section 5.6 when we introduce *persistence*, a generalization of the notion of canonicity, and prove the Sahlqvist Completeness Theorem.

Exercises for Section 5.3

5.3.1 Prove that for any frame $\mathfrak{F}$, $\mathfrak{ue}\,\mathfrak{F} = (\mathfrak{F}^+)_+$.

5.3.2 Let Λ be a normal modal logic. Give a detailed proof that the canonical frame $\mathfrak{F}^\Lambda$ is isomorphic to the canonical extension of $\mathfrak{L}_\Lambda$.

5.3.3 Let A denote the collection of sets X of integers satisfying one of the following four conditions: (i) X is finite, (ii) X is co-finite, (iii) $X \oplus E$ is finite, (iv) $X \oplus E$ is co-finite. Here E denotes the set of all even integers, and $\oplus$ denotes symmetric difference: $X \oplus E = (X \setminus E) \cup (E \setminus X)$. Consider the following algebra $\mathfrak{A} = (A, \cup, -, \varnothing, f)$ where the operation f is given by

$$f(X) = \begin{cases} \{x - 1 \mid x \in X\} & \text{if } X \text{ is of type (i) or (iii),} \\ \mathbb{Z} & \text{if } X \text{ is of type (ii) or (iv).} \end{cases}$$

(a) Show that $\mathfrak{A}$ is a boolean algebra with operators.
(b) Describe $\mathfrak{A}_+$.

5.3.4 Let W be the set $\mathbb{Z} \cup \{-\infty, \infty\}$ and let S be the successor relation on $\mathbb{Z}$, that is, $S = \{(z, z+1) \mid z \in \mathbb{Z}\}$.

(a) Give a BAO whose ultrafilter frame is isomorphic to the frame $\mathfrak{F} = (W, R)$ with $R = S \cup \{(-\infty, -\infty), (\infty, \infty)\}$.
(b) Give a BAO whose ultrafilter frame is isomorphic to the frame $\mathfrak{F} = (W, R)$ with $R = S \cup (W \times \{-\infty, \infty\})$.
(c) Give a BAO whose ultrafilter frame is isomorphic to the frame $\mathfrak{F} = (W, R)$ with $R = S \cup \{(-\infty, -\infty)\} \cup (W \times \{\infty\})$.

5.3.5 An operation on a boolean algebra is called *2-additive* if it satisfies

$$f(x + y + z) = f(x + y) + f(x + z) + f(y + z).$$

Now suppose that $\mathfrak{A} = (A, +, -, 0, f)$ such that $(A, +, -, 0)$ is a boolean algebra on which f is a 2-additive operation. Prove that this algebra can be embedded in a complete and atomic such algebra.

5.4 Duality Theory

We now know how to build frames from algebras and algebras from frames in ways that preserve crucial logical properties. But something is missing. Modal logicians rarely study frames in isolation: rather, they are interested in how to construct new frames from old using bounded morphisms, generated subframes, and disjoint unions. And algebraists adopt an analogous perspective: they are interested in relating algebras via such constructions as homomorphisms, subalgebras, and direct products. Thus modal logicians work in one mathematical universe, and algebraists in another, and it is natural to ask whether these universes are systematically related. They are, and duality theory studies these links.

In this section we will do two things. First, we will introduce the basic dualities that exist between the modal and algebraic universes. Second, we will demonstrate that these dualities are useful by proving two major theorems of modal logic. We assume that by this stage the reader has picked up the basic definitions and results concerning the algebraic universe (and in particular, what homomorphisms, subalgebras, and direct products are). If not, check out Appendix B.

Basic duality results

Theorems 5.47 and 5.48 below give a concise formulation of the basic links between the algebraic and frame-theoretic universes. They are stated using the following notation.

Definition 5.46 Let τ be a modal similarity type, $\mathfrak{F}$ and $\mathfrak{G}$ two τ-frames, and $\mathfrak{A}$ and $\mathfrak{B}$ two boolean algebras with τ-operators. We recall (define, respectively) the following notation for relations between these structures:

- $\mathfrak{F} \rightarrowtail \mathfrak{G}$ for $\mathfrak{F}$ is isomorphic to a generated subframe of $\mathfrak{G}$,
- $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$ for $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$,
- $\mathfrak{A} \rightarrowtail \mathfrak{B}$ for $\mathfrak{A}$ is isomorphic to a subalgebra of $\mathfrak{B}$,
- $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$ for $\mathfrak{B}$ is a homomorphic image of $\mathfrak{A}$. ⊣

Theorem 5.47 *Let τ be a modal similarity type, $\mathfrak{F}$ and $\mathfrak{G}$ two τ-frames, and $\mathfrak{A}$ and $\mathfrak{B}$ two boolean algebras with τ-operators.*

(i) *If $\mathfrak{F} \rightarrowtail \mathfrak{G}$, then $\mathfrak{G}^+ \twoheadrightarrow \mathfrak{F}^+$.*
(ii) *If $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$, then $\mathfrak{G}^+ \rightarrowtail \mathfrak{F}^+$.*
(iii) *If $\mathfrak{A} \rightarrowtail \mathfrak{B}$, then $\mathfrak{B}_+ \twoheadrightarrow \mathfrak{A}_+$.*
(iv) *If $\mathfrak{A} \twoheadrightarrow \mathfrak{B}$, then $\mathfrak{B}_+ \rightarrowtail \mathfrak{A}_+$.*

Proof. This follows immediately from Propositions 5.51 and 5.52 below. ⊣

Theorem 5.48 *Let τ be a modal similarity type, and $\mathfrak{F}_i, i \in I$, a family of τ-frames. Then*

$$\left(\biguplus_{i\in I}\mathfrak{F}_i\right)^+ \cong \prod_{i\in I}\mathfrak{F}_i^+.$$

Proof. We define a map η from the power set of the disjoint union $\biguplus_{i\in I} W_i$ to the carrier $\prod_{i\in I}\mathcal{P}(W_i)$ of the product of the family of complex algebras $(\mathfrak{F}_i^+)_{i\in I}$.

Let X be a subset of $\biguplus_{i\in I} W_i$; $\eta(X)$ has to be an element of the set $\prod_{i\in I}\mathcal{P}(W_i)$. And elements of the set $\prod_{i\in I}\mathcal{P}(W_i)$ are sequences σ such that $\sigma(i) \in \mathcal{P}(W_i)$. So it suffices to say what the i-th element of the sequence $\eta(X)$ is:

$$\eta(X)(i) = X \cap W_i.$$

We leave it as an exercise to show that η is an isomorphism; see Exercise 5.4.6. ⊣

Note that Theorem 5.48 (in contrast to Theorem 5.47) only states a connection in the direction from frames to algebras. This is because in general

$$\left(\prod_{i\in I}\mathfrak{A}_i\right)_+ \not\cong \biguplus_{i\in I}(\mathfrak{A}_i)_+.$$

The reader is asked to give an example to this effect in Exercise 5.4.1.

In order to prove Theorem 5.47, the reader is advised to recall the definitions of the back and forth properties of bounded morphisms between frames (Definition 3.13). We also need some terminology for morphisms between boolean algebras with operators.

Definition 5.49 Let $\mathfrak{A}$ and $\mathfrak{A}'$ be two BAOs of the same similarity type, and let $\eta : A \to A'$ be a function. We say that η is a *boolean homomorphism* if η is a homomorphism from $(A, +, -, 0)$ to $(A', +', -', 0')$. We call η a *modal homomorphism* if η satisfies, for all modal operators $\triangle$:

$$\eta(f_\triangle(a_1, \ldots, a_{\rho(\triangle)})) = f'_\triangle(\eta a_1, \ldots, \eta a_{\rho(\triangle)}).$$

(Here ηa_i means $\eta(a_i)$; we will sometimes use this shorthand to keep the notation uncluttered.) Finally, η is a *(*BAO*-)homomorphism* if it is both a boolean and a modal homomorphism. ⊣

In the following definition, the construction of *dual* or *lifted* morphisms is given (here the word 'dual' is *not* used in the sense of $\Diamond$ being the dual of $\Box$).

Definition 5.50 Suppose θ is a map from W to W'; then its *dual*, $\theta^+ : \mathcal{P}(W') \to \mathcal{P}(W)$ is defined as:

$$\theta^+(X') = \{u \in W \mid \theta(u) \in X'\}.$$

In the other direction, let $\mathfrak{A}$ and $\mathfrak{A}'$ be two BAOs, and $\eta : \mathfrak{A} \to \mathfrak{A}'$ be a map from A to A'; then its *dual* is given as the following map from ultrafilters of $\mathfrak{A}$ to subsets of A:

$$\eta_+(u') = \{a \in A \mid \eta(a) \in u'\}. \quad \dashv$$

The following propositions assert that the duals of bounded morphisms are nothing but BAO-homomorphisms:

Proposition 5.51 *Let $\mathfrak{F}$, $\mathfrak{F}'$ be frames, and $\theta : W \to W'$ a map.*

(i) θ^+ *is a boolean homomorphism.*
(ii) $m_R(\theta^+(Y_1'), \ldots, \theta^+(Y_n')) \subseteq \theta^+(m_{R'}(Y_1', \ldots, Y_n'))$, *if θ has the forth property.*
(iii) $m_R(\theta^+(Y_1'), \ldots, \theta^+(Y_n')) \supseteq \theta^+ m_{R'}(Y_1', \ldots, Y_n'))$, *if θ has the back property.*
(iv) θ^+ *is a* BAO*-homomorphism from $\mathfrak{F}^+$ to $\mathfrak{F}'^+$, if θ is a bounded morphism.*
(v) θ^+ *is surjective,if θ is injective.*
(vi) θ^+ *is injective, if θ is surjective.*

Proof. For notational convenience, we assume that τ has only one modal operator, so that we can write $\mathfrak{F} = (W, R)$.

(i). (Note that this was Exercise 5.1.1.) As an example, we treat complementation:

$$x \in \theta^+(-X') \text{ iff } \theta(x) \in -X' \text{ iff } \theta(x) \notin X' \text{ iff } x \notin \theta^+(X').$$

From this it follows immediately that $\theta^+(-X') = -\theta^+(X')$.

(ii). Assume that θ has the forth property. Then we have

$$\begin{aligned}
& x \in m_R(\theta^+(Y_1'), \ldots, \theta^+(Y_n')) \\
& \quad \Longrightarrow \ \exists\, y_1, \ldots, y_n \text{ such that } \theta(y_i) \in Y_i' \text{ and } Rxy_1 \ldots y_n \\
& \quad \Longrightarrow \ \exists\, y_1, \ldots, y_n \text{ such that } \theta(y_i) \in Y_i' \text{ and } R'\theta(x)\theta(y_1) \ldots \theta(y_n) \\
& \quad \Longrightarrow \ \theta(x) \in m_{R'}(Y_1', \ldots, Y_n') \\
& \quad \Longrightarrow \ x \in \theta^+(m_{R'}(Y_1', \ldots, Y_n')).
\end{aligned}$$

(iii). Now suppose $x \in \theta^+(m_{R'}(Y_1', \ldots, Y_n'))$. Then $\theta(x) \in m_{R'}(Y_1', \ldots, Y_n')$. So there are $y_1', \ldots, y_n'$ in W' with $y_i' \in Y_i'$ and $R'\theta(x)y_1' \ldots y_n'$. As θ has the back property, there are $y_1, \ldots, y_n \in W$ with $\theta(y_i) = y_i'$ for all i, and $Rxy_1 \ldots y_n$. But then $y_i \in \theta^+(Y_i')$ for every i, so $x \in m_R(\theta^+(Y_1'), \ldots, \theta^+(Y_n'))$.

(iv). This follows immediately from items (i), (ii) and (iii).

(v). Assume that θ is injective, and let X be a subset of W. We have to find a subset X' of W' such that $\theta^+(X') = X$. Define

$$\theta[X] := \{\theta(x) \in W' \mid x \in X\}.$$

We claim that this set has the desired properties. Clearly $X \subseteq \theta^+(\theta[X])$. For the other direction, let x be an element of $\theta^+(\theta[X])$. Then by definition, $\theta(x) \in \theta[X]$, so there is a $y \in X$ such that $\theta(x) = \theta(y)$. By the injectivity of θ, $x = y$. So $x \in X$.

(vi). Assume that θ is surjective, and let X' and Y' be distinct subsets of W'. Without loss of generality we may assume that there is an x' such that $x' \in X'$ and $x' \notin Y'$. As θ is surjective, there is an x in W such that $\theta(x) = x'$. So $x \in \theta^+(X')$, but $x \notin \theta^+(Y')$. So $\theta(X') \neq \theta(Y')$, whence θ^+ is injective. ⊣

Going in the opposite direction, that is, from algebras to relational structures, we find that the duals of BAO-homomorphisms are bounded morphisms:

Proposition 5.52 *Let $\mathfrak{A}, \mathfrak{A}'$ be boolean algebras with operators, and η a map from A to A'.*

(i) *If η is a boolean homomorphism, then η_+ maps ultrafilters to ultrafilters.*
(ii) *If $f'(\eta(a_1), \ldots, \eta(a_n)) \leq \eta(f(a_1, \ldots, a_n))$, then η_+ has the forth property.*
(iii) *If $f'(\eta(a_1), \ldots, \eta(a_n)) \geq \eta(f(a_1, \ldots, a_n))$ and η is a boolean homomorphism, then η_+ has the back property.*
(iv) *If η is a BAO-homomorphism, then η_+ is a bounded morphism from $\mathfrak{A}'_+$ to $\mathfrak{A}_+$.*
(v) *If η is an injective boolean homomorphism, then $\eta_+ : Uf\mathfrak{A}' \to Uf\mathfrak{A}$ is surjective.*
(vi) *If η is an surjective boolean homomorphism, then $\eta_+ : Uf\mathfrak{A}' \to Uf\mathfrak{A}$ is injective.*

Proof. Again, without loss of generality we assume that τ has only one modal operator, so that we can write $\mathfrak{A} = (A, +, -, 0, f)$.

(i). This item is left as Exercise 5.4.2.

(ii). Suppose that $Q_{f'}u'u'_1 \ldots u'_n$ holds between some ultrafilters $u', u'_1, \ldots, u'_n$ of $\mathfrak{A}'$. To show that $\mathfrak{A}_+ \models Q_f\eta_+u'\eta_+u'_1 \ldots \eta_+u'_n$, let $a_1, \ldots, a_n$ be arbitrary elements of $\eta_+u'_1, \ldots, \eta_+u'_n$ respectively. Then, by definition of η_+, $\eta a_i \in u'_i$, so $Q'_f u'u'_1 \ldots u'_n$ gives $f'(\eta a_1, \ldots, \eta a_n) \in u'$. Now the assumption yields $\eta f(a_1, \ldots, a_n) \in u'$, as ultrafilters are upward closed. But then $f(a_1, \ldots, a_n) \in \eta_+ u'$, which is what we wanted.

(iii). This item is left as Exercise 5.4.2.

(iv). This follows immediately from items (i), (ii) and (iii).

(v). Assume that η is injective, and let u be an ultrafilter of $\mathfrak{A}$. We want to follow the same strategy as in Proposition 5.51(v), and define

$$\eta[u] := \{\eta(a) \mid a \in u\}.$$

The difference with the proof of Proposition 5.51(v) is that here, $\eta_+(\eta[u])$ may not be defined. The reason for this is that, in general, $\eta[u]$ will not be upwards closed and hence, not an (ultra)filter, while η_+ is defined only for ultrafilters. Therefore, we define

$$F' := \{a' \mid \eta(a) \leq a' \text{ for some } a \in u\}.$$

Clearly, $\eta[u] \subseteq F'$. We will first show that F' is a *proper filter* of $\mathfrak{A}'$ (note that the clauses $(F1)$–$(F3)$ which define filters are given in Definition 5.34). For $(F1)$, observe that $1 \in u$, so $\eta(1) = 1 \in \eta[u] \subseteq F'$. For $(F2)$, assume $a', b' \in F'$. Then there are a, b in u such that $\eta a \leq a'$ and $\eta b \leq b'$. It follows that $\eta(a \cdot b) = \eta a \cdot \eta b \leq a' \cdot b' \in \eta[u]$; hence, $a' \cdot b' \in F'$ since $a \cdot b \in u$. This shows that F' is closed under taking meets. It is trivial to prove $(F3)$, that is, that F' is upwards closed. Finally, in order to show that F' is proper, suppose that $0' \in F'$. Then $0' = \eta a$ for some $a \in u$; as $0' = \eta(0)$, injectivity of η gives that $0 = a$, and hence, $0 \in u$. But then u is not an ultrafilter.

By the Ultrafilter Theorem 5.38, F' can be extended to an ultrafilter u'. We claim that $u = \eta_+(u')$. First let a be in u, then $\eta a \in \eta[u] \subseteq u'$, so $a \in \eta_+(u')$. This shows that $u \subseteq \eta_+(u')$. For the other inclusion, it suffices to show that $a \notin \eta_+(u')$ if $a \notin u$; we reason as follows:

$$\begin{array}{rcl} a \notin u & \Longrightarrow & -a \in u \\ & \Longrightarrow & -\eta a = \eta(-a) \in \eta[u] \\ & \Longrightarrow & -\eta(a) \in u' \\ & \Longrightarrow & \eta a \notin u' \\ & \Longrightarrow & a \notin \eta_+(u') \end{array}$$

(vi). Similar to Proposition 5.51, item (vi); see Exercise 5.4.2. ⊣

Readers familiar with category theory will have noticed that the operation $(\cdot)^+$ is a functor from the category of τ-frames with bounded morphisms to the category of boolean algebras with τ-operators, and vice versa for $(\cdot)_+$. This categorial perspective is implicit in what follows, but seldom comes to the surface. In the remainder of the section we will see how our algebraic perspective on modal logic that we have developed can be applied.

Applications

In this subsection we tie a number of threads together and show how to use the duality between frames and algebras to give very short proofs of some major theorems of modal logic.

Our first example shows that all the results given in Proposition 3.14 on the preservation of *modal* validity under the fundamental frame operations fall out as simple consequences of well-known preservation results of universal algebra, namely that *equational* validity is preserved under the formation of subalgebras, homomorphic images and products of algebras.

Proposition 5.53 *Let τ be a modal similarity type, ϕ a τ-formula and $\mathfrak{F}$ a τ-frame. Then*

(i) *If $\mathfrak{G}$ is a bounded morphic image of $\mathfrak{F}$, then $\mathfrak{G} \Vdash \phi$ if $\mathfrak{F} \Vdash \phi$.*
(ii) *If $\mathfrak{G}$ is a generated subframe of $\mathfrak{F}$, then $\mathfrak{G} \Vdash \phi$ if $\mathfrak{F} \Vdash \phi$.*
(iii) *If $\mathfrak{F}$ is the disjoint union of a family $\{\mathfrak{F}_i \mid i \in I\}$, then $\mathfrak{F} \Vdash \phi$ if for every $i \in I$, $\mathfrak{F}_i \Vdash \phi$.*
(iv) *If $\mathfrak{ue}\,\mathfrak{F} \Vdash \phi$ then $\mathfrak{F} \Vdash \phi$.*

Proof. We only prove the first part of the proposition, leaving the other parts as exercises for the reader.

Assume that $\mathfrak{F} \twoheadrightarrow \mathfrak{G}$, and $\mathfrak{F} \Vdash \phi$. By Proposition 5.24, we have $\mathfrak{F}^+ \models \phi \approx \top$, and by Theorem 5.47, $\mathfrak{G}^+$ is a *subalgebra* of $\mathfrak{F}^+$. So by the fact that equational validity is preserved under taking subalgebras, we obtain that $\phi \approx \top$ holds in $\mathfrak{G}^+$. But then Proposition 5.24 implies that $\mathfrak{G} \Vdash \phi$. ⊣

Our second example is a simple proof of the Goldblatt-Thomason Theorem, which gives a precise structural characterization of the first-order definable classes of frames which are modally definable. We discussed this result in Chapter 3, and gave a proof which drew on the tools of first-order model theory (see Theorem 3.19 in Section 3.8). As we will now see, there is also an algebraic way of viewing the theorem: it is a more or less immediate corollary of Birkhoff's Theorem (see Appendix B) identifying equational classes and varieties. The version we prove here is slightly stronger than Theorem 3.19, since it applies to any class of frames that is closed under taking ultrapowers.

Theorem 5.54 (Goldblatt-Thomason Theorem) *Let τ be a modal similarity type, and let K be a class of τ-frames that is closed under taking ultrapowers. Then K is modally definable if and only if it is closed under the formation of bounded morphic images, generated subframes, and disjoint unions, and reflects ultrafilter extensions.*

Proof. The left to right direction is an immediate corollary of the previous proposition. For the right to left direction, let K be any class of frames satisfying the closure conditions given in the theorem. It suffices to show that any frame $\mathfrak{F}$ validating the modal theory of K is itself a member of K.

Let $\mathfrak{F}$ be such a frame. It is not difficult to show that Proposition 5.24 implies that $\mathfrak{F}^+$ is a model for the equational theory of the class $\mathbf{Cm}\mathsf{K}$. It follows by Birkhoff's Theorem (identifying varieties and equational classes) that $\mathfrak{F}^+$ is in the variety generated by $\mathbf{Cm}\mathsf{K}$, so $\mathfrak{F}^+$ is in $\mathbf{HSPCm}\mathsf{K}$. In other words, there is a family $(\mathfrak{G}_i)_{i\in I}$ of frames in K, and there are boolean algebras with operators $\mathfrak{A}$ and $\mathfrak{B}$ such that

(i) $\mathfrak{B}$ is the product $\prod_{i\in I}\mathfrak{G}_i^+$ of the complex algebras of the $\mathfrak{G}_i$,
(ii) $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$, and
(iii) $\mathfrak{F}^+$ is a homomorphic image of $\mathfrak{A}$.

By Theorem 5.48, $\mathfrak{B}$ is isomorphic to the complex algebra of the disjoint union $\mathfrak{G}$ of the family $(\mathfrak{G}_i)_{i\in I}$:

$$\mathfrak{B} \cong \mathfrak{G}^+ = \left(\biguplus_{i\in I}\mathfrak{G}_i\right)^+.$$

As K is closed under taking disjoint unions, $\mathfrak{G}$ is in K.

Now we have the following picture: $\mathfrak{F}^+ \twoheadleftarrow \mathfrak{A} \rightarrowtail \mathfrak{G}^+$. By Theorem 5.47 it follows that

$$(\mathfrak{F}^+)_+ \rightarrowtail \mathfrak{A}_+ \twoheadleftarrow (\mathfrak{G}^+)_+.$$

Since K is closed under ultrapowers, Theorem 3.17 implies that $(\mathfrak{G}^+)_+ = \mathfrak{ue}\,\mathfrak{G}$ is in K. As K is closed under the formation of bounded morphic images and generated subframes, it follows that $\mathfrak{A}_+$ and $\mathfrak{ue}\,\mathfrak{F} = (\mathfrak{F}^+)_+$ (in that order) are in K. But then $\mathfrak{F}$ itself is also a member of K, since K reflects ultrafilter extensions. ⊣

For our third example, we return to the concept of canonicity. We will prove an important result and mention an intriguing open problem, both having to do with the relation between canonical varieties and first-order definable classes of frames. Both the result and the open problem were mentioned in Chapter 4 (see Theorem 4.50 and the surrounding discussion), albeit in a slightly weaker form. To link the earlier statements with the versions discussed here, simply observe that any elementary class of frames is closed under the formation of ultraproducts.

First we need the following definition.

Definition 5.55 Let τ be modal similarity type, and K be a class of τ-frames. The variety generated by K (notation: $\mathbf{V}_\mathsf{K}$) is the class $\mathbf{HSPCm}\mathsf{K}$. ⊣

Theorem 5.56 *Let τ be modal similarity type, and K be a class of τ-frames which is closed under ultraproducts. Then the variety $\mathbf{V}_\mathsf{K}$ is canonical.*

Proof. Assume that the class K of τ-frames is closed under taking ultraproducts. We will first prove that the class $\mathbf{HSCm}\mathsf{K}$ is canonical. Let $\mathfrak{A}$ be an element of this class; that is, assume that there is a frame $\mathfrak{F}$ in K and an algebra $\mathfrak{B}$ such that

$$\mathfrak{A} \twoheadleftarrow \mathfrak{B} \rightarrowtail \mathfrak{F}^+.$$

It follows from Theorem 5.47 that

$$\mathfrak{Em}\mathfrak{A} \twoheadleftarrow \mathfrak{Em}\mathfrak{B} \rightarrowtail \mathfrak{Em}\mathfrak{F}^+ = (\mathfrak{ue}\,\mathfrak{F})^+. \tag{5.27}$$

From Theorem 3.17 we know that $\mathfrak{ue}\,\mathfrak{F}$ is the bounded morphic image of some ultrapower $\mathfrak{G}$ of $\mathfrak{F}$. Note that $\mathfrak{G}$ is in K, by assumption. Now Theorem 5.47 gives

$$(\mathfrak{ue}\,\mathfrak{F})^+ \rightarrowtail \mathfrak{G}^+. \tag{5.28}$$

Since $\mathfrak{G}^+$ is in $\mathbf{Cm}\mathsf{K}$, (5.27) and (5.28) together imply that $\mathfrak{A}$ is in $\mathbf{HSCm}\mathsf{K}$. Hence this class is canonical.

To prove that the *variety* generated by K is canonical, we need an additional fact. Recall that according to Proposition 3.63, the ultrapower of a disjoint union can be obtained as the disjoint union of ultraproducts.

Now assume that $\mathfrak{A}$ is in $\mathbf{V}_{\mathsf{K}} = \mathbf{HSPCm}\mathsf{K}$. In other words, assume there is a family $\{\mathfrak{F}_i \mid i \in I\}$ of frames in K and an algebra $\mathfrak{B}$ such that

$$\mathfrak{A} \twoheadleftarrow \mathfrak{B} \rightarrowtail \prod_{i\in I} \mathfrak{F}_i^+.$$

To prove that $\mathfrak{Em}\mathfrak{A}$ is in $\mathbf{V}_{\mathsf{K}}$, it suffices to show that $\mathfrak{Em}(\prod_{i\in I}\mathfrak{F}_i^+)$ is in $\mathbf{SPCm}\mathsf{K}$ — the remainder of the proof is as before. Let $\mathfrak{F}$ be the frame $\biguplus_{i\in I}\mathfrak{F}_i$, then by Theorem 5.48, $\mathfrak{F}^+ \cong \prod_{i\in I}\mathfrak{F}_i^+$. Hence, by Theorem 5.47:

$$\mathfrak{Em}\left(\prod_{i\in I}\mathfrak{F}_i^+\right) \cong ((\mathfrak{F}^+)_+)^+ = (\mathfrak{ue}\,\mathfrak{F})^+. \tag{5.29}$$

By Theorem 3.17, there is an ultrapower $\mathfrak{G}$ of $\mathfrak{F}$ such that $\mathfrak{G} \twoheadrightarrow \mathfrak{ue}\,\mathfrak{F}$. Now we apply Proposition 3.63, yielding a frame $\mathfrak{H}$ such that (i) $\mathfrak{H}$ is a disjoint union of ultraproducts of frames in K and (ii) $\mathfrak{H} \twoheadrightarrow \mathfrak{G}$. Putting these observations together we have $\mathfrak{ue}\,\mathfrak{F} \twoheadleftarrow \mathfrak{G} \twoheadleftarrow \mathfrak{H}$. Hence, by Theorem 5.47:

$$(\mathfrak{ue}\,\mathfrak{F})^+ \rightarrowtail \mathfrak{G}^+ \rightarrowtail \mathfrak{H}^+. \tag{5.30}$$

Note that $\mathfrak{H}$ is a disjoint union of frames in K, since K is closed under taking ultraproducts. This implies that $\mathfrak{H}^+$ is in $\mathbf{PCm}\mathsf{K}$. But then it follows from (5.29) and (5.30) that $\mathfrak{Em}(\prod_{i\in I}\mathfrak{F}_i^+)$ is in $\mathbf{SPCm}\mathsf{K}$, which is what we needed. ⊣

Example 5.57 Consider the modal similarity type $\{\circ, \otimes, 1'\}$ of arrow logic, where $\circ$ is binary, $\otimes$ is unary and 1' is a constant. The standard interpretation of this language is given in terms of the *squares* (cf. Example 1.24). Recall that the square $\mathfrak{S}_U = (W, C, R, I)$ is defined as follows.

$$\begin{array}{rcl} W & = & U \times U \\ C((u,v),(w,x),(y,z)) & \text{iff} & u = w \text{ and } v = z \text{ and } x = y \\ R((u,v),(w,x)) & \text{iff} & u = x \text{ and } v = w \end{array}$$

$$I(u, v) \quad \text{iff} \quad u = v.$$

It may be shown that the class SQ of (isomorphic copies of) squares is first-order definable in the frame language with predicates C, R and I. Therefore, Theorem 5.56 implies that the variety generated by SQ is canonical. This variety is well-known in the literature on algebraic logic as the variety RRA of Representable Relation Algebras. See Exercise 5.4.5. ⊣

Rephrased in terminology from modal logic, Theorem 5.56 boils down to the following result.

Corollary 5.58 *Let τ be a modal similarity type, and K be a class of τ-frames which is closed under ultraproducts. Then the modal theory of K is a canonical logic.*

We conclude the section with the foremost open problem in this area: does the converse of Theorem 5.56 holds as well?

Open Problem 2 *Let τ be modal similarity type, and V a canonical variety of boolean algebras with τ-operators. Is there a class K of τ-frames, closed under taking ultraproducts, such that V is generated by K?*

Exercises for Section 5.4

5.4.1 Consider a countably infinite collection $(\mathfrak{A}_i)_{i \in I}$ of finite algebras that are non-trivial, that is, of size at least 2.

(a) Show that the product $\prod_{i \in I} \mathfrak{A}_i$ has uncountably many ultrafilters.
(b) Show that the ultrafilter frame of a finite algebra is finite, and that hence, the disjoint union $\biguplus_{i \in I} (\mathfrak{A}_i)_+$ is countable.
(c) Conclude that $\left(\prod_{i \in I} \mathfrak{A}_i\right)_+ \not\cong \biguplus_{i \in I} (\mathfrak{A}_i)_+$.

5.4.2 Prove Proposition 5.52(i), (iii) and (vi). Prove (iii) first for unary operators; for the general case, see the proof of the Jónsson-Tarski theorem for inspiration.

5.4.3 Prove or disprove the following propositions:

(a) For any two boolean algebras with operators $\mathfrak{A}$ and $\mathfrak{B}$: $\mathfrak{A}_+ \cong \mathfrak{B}_+$ only if $\mathfrak{A} \cong \mathfrak{B}$. (Hint: first consider the question for plain boolean algebras, and then consider specimens of BAOs as in Exercise 5.2.3 and Exercise 5.3.3.)
(b) For any two frames $\mathfrak{F}$ and $\mathfrak{G}$: $\mathfrak{F}^+ \cong \mathfrak{G}^+$ only if $\mathfrak{F} \cong \mathfrak{G}$.

5.4.4 Consider the frames $\mathfrak{F} = (X, R)$ and $\mathfrak{G} = (Y, S)$ given by

$$X = \mathbb{N} \qquad Y = \mathbb{N} \cup \{\infty\}$$
$$R = \{(x, y) \in X \times X \mid x \neq y\} \qquad S = \{(x, y) \in Y \times Y \mid x \neq y\} \cup \{(\infty, \infty)\}$$

(a) Show that $\mathfrak{F}$ is not a bounded morphic image of $\mathfrak{G}$.